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Approximate Orthogonal Preserving Mappings on C^*-Semi-inner Product Space

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In this research work, we study Birkhoff-James orthogonality in an arbitrary normed linear space X, and establish the orthogonal preserving mapping in C^*-semi-inner product space. It has been observed that every mapping that preserves orthogonality is necessarily a scalar multiple of an isometry. We finally introduce approximate orthogonality and investigate this notion in C^*-semi-inner product space.

Keywords: C^*-semi-inner product, Birkhoff-James orthogonality, Approximate orthogonality.

1. Introduction

The semi-inner product spaces were first seen in the work of Lumer [1], where he put into consideration some vector spaces and defined a mapping on them which is linear in one argument only, strictly positive and satisfies Cauchy-Schwarz's inequality. Giles [2] studied properties on Hilbert space in relation to the theory of Banach space where fundamental properties and consequences of semi-inner product spaces were established. Giles [2] furthermore described structural modifications to the concept of semi-inner product by striking homogeneity property producing much convenience without causing any significant limitation.

A generalization of the concept of semi-inner product space introduced in [1] has also been considered by Nath [3], where Cauchy-Schwarz's inequality is replaced by Holder's inequality.

C^*-Semi-inner product spaces are new generalization of Hilbert C^*-modules connecting Hilbert C^*-modules and Finsler modules. The C^*-semi-inner product spaces naturally generalize semi-inner product spaces by letting semi-inner products to take values in an arbitrary C^*-algebra instead of the C^*-algebra of complex numbers, which attest its usefulness both theoretically and practically as can be seen in [4].

On the other hand, in a normed linear space there is various conception of orthogonality, which generalizes the orthogonality in an inner product space. However, one of the most momentous is the concept of Birkhoff-James orthogonality which undertake a central role in approximation theory.

Remark 1.1

An inner product A-module (where A is the underlying C^*-algebra) is a complex vector space E such that E is a right A-module satisfying the following:

1. $\langle x, \lambda y + z\rangle = \lambda\langle x, y\rangle + \langle x, z\rangle$

2. $\langle x, y.a\rangle = \langle x, y\rangle a$

3. $\langle x, y\rangle^* = \langle y, x\rangle$

4. $\langle x, x\rangle > 0$

5. $\langle x, x\rangle = 0 \Rightarrow x = 0 \quad (x, y, z \in E, a \in A, \lambda \in \mathbb{C})$,

If only conditions (1) to (4) hold, E is then called a semi-inner product A-module, and $[.,.]$ denote the semi-inner product on E.

Elements $x \text{ and } y$ in a semi-inner product A-module E are said to be orthogonal if $[x, y] = 0$ and, for a given $\varepsilon \in [0,1)$ they are approximately orthogonal or ε-orthogonal if $\|[x, y]\| \leq \varepsilon\|x\|\|y\|$

A mapping $T: X \to Y$, where X and Y are semi-inner product A-module, preserves orthogonality if $[x, y] = 0$ implies $[Tx, Ty] = 0$ for all $x, y \in X$, and consequently it is an approximate orthogonal preserving mapping if $[x, y] = 0$ implies that $\|[Tx, Ty]\| \leq \varepsilon \|Tx\| \|Ty\|$

In the present research work we tend to investigate the approximate orthogonal preserving mapping in C^*-semi-inner product space.

2. Preliminaries

Definition 2.1

Banach Algebras and C^* - Algebras

We shall use the term algebra to mean linear associative algebra. If no field is mentioned, the scalars will be the complex field.

Let F be a field, and let A be a vector space over F equipped with an additional binary operation from A × A to A, denoted here by ·, then A is an algebra over F if the following identities hold for any three elements x, y, and z of A, and all scalars a and b of F:

i. $(x + y) \cdot z = x \cdot z + y \cdot z$
ii. $x \cdot (y + z) = x \cdot y + x \cdot z$
iii. $(ax) \cdot (by) = (ab)(x \cdot y)$

These three axioms are another way of saying that the binary operation is bilinear. An algebra over A is sometimes also called a F-algebra, and F is called the base field of A as can be seen in [5].

Definition 2.2

An algebra A is said to be a normed algebra if it has a norm that makes it into a normed linear space and the norm also satisfies

I. $\|ab\| \leq \|a\| \|b\| \quad a, b \in A$
II. If A has an identity e then $\|e\| = 1.$

If A is normed and is a Banach space then it is called a Banach Algebra [6].

Definition 2.3

An involution on A is a map $*: A \longrightarrow A(a \longmapsto a^*)$ such that

1. $a^{**} = a \;\; a \in A$
2. $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\lambda} b^* \lambda \in \mathbb{F}$
3. $(ab)^* = b^* a^*. \;\; a, b \in A$

If A is a Banach algebra with involution and also

4. $\|aa^*\| = \|a\|^2$

Then A is called a C^* - algebra and (4) is called the C^* - "condition".

Definition 2.4

Let A be a C^*algebra. A semi-inner product A-module is a linear space E which is a right A-module with compatible scalar multiplication $\lambda(xa) = (\lambda x)a = x(\lambda a)$ together with a map $[x, y]: E \times E \to A$ such that for all $x \in E, a \in A, \lambda \in \mathbb{C}$ the following properties are satisfied:

i $[x, x] \geq 0$. and $[x, x] = 0 \Rightarrow x = 0$

ii $[x, \alpha y + \beta z] = \alpha[x, y] + \beta[x, z]$.

iii $[x, y]^* = [y, x]$.

iv $[x, ya] = [x, y]a$ and $[xa, y] = a^*[x, y]$.

v $\|[x, y]\|^2 \leq \|[x, x]\| \|[y, y]\|$.

The triple $(E, A, [., .])$ is called a C^*-semi-inner product space or E is a semi-inner product A-module [4].

Example 2.5

Let $(X_i [., .]_i)$ be a semi-inner product A_i-module, $1 \leq i \leq n$. If for $(a_1, \dots, a_n) \in A$ and $(x_1, \dots, x_n) \in \bigoplus_{i=1}^{n} X_i$,

We define,

$(x_1, \dots, x_n)(a_1, \dots, a_n) = (x_1 a_1, \dots, x_n a_n)$

And,

the C^*-semi-inner product is defined as follows:

$[(x_1, \dots, x_n), (y_1, \dots, y_n)] = ([x_1, y_1]_1, \dots, [x_n, y_n]_n)$, Then,

($X_i [.,.]_i$) is a semi-inner product A-module.

3. Orthogonality in C^*-semi-inner product in Birkhoff-James sense

In this section we plan to study the relations between Birkhoff-James orthogonality and the orthogonality in a C^*-semi-inner product space. We then extend some of the already known results on the orthogonality in C^*-semi-inner product spaces to establish the orthogonal

preserving mappings in the setting of semi-inner product A-modules over an arbitrary C^*-algebra A.

Remark 3.1

A semi-inner product A-module V has an A-valued "norm" in [7] given by $|x| = [x,x]^{\frac{1}{2}}$. Since taking square root of a positive element is an order-preserving operation in a C^*-algebra, it follows that $\|[x,y]\| \leq \|x\| \, |y|$ for all $x, y \in V$.

Also for all $a \in A$, $|a| = (a^*a)^{\frac{1}{2}}$ and we note that the norm on V makes V into a normed A-module. i.e. $\|xa\| \leq \|x\| \, \|a\|$.

A vector x in a normed space $(E, \|.\|)$ is said to be orthogonal in the Birkhoff-James sense to a vector $y \in E$ if $\|x\| \leq \|x + \lambda y\| \; \forall$ λ, [8,9]. For inner product spaces, this definition is identical to the frequent definition of orthogonality. In the following proposition we plan to establish this fact to semi- inner product A-modules, taking into account the role of scalars is now played by the elements of the C^*-algebra A.

Proposition 3.2

Let A be a C^*-algebra and let V be a semi- inner product A-module. For $x, y \in V$, the following conditions are equivalent:

(i) $[x,y]$= 0

$|x|^2 \leq |x + ay|^2 \;\; a \in A$

Proof:

(i) ⇒ (ii). If $[x,y]$= 0, then

$$|x|^2 \leq |x + ay|^2 = [x + ay, x + ay]$$

$$\leq |x|^2 + a^*[x,y] + a[y,x] + |ay|^2$$

$$|x|^2 \leq |x|^2 + ay^2 = |x + ay|^2 \;\; \forall a \in A.$$

(ii) ⟹ (i) we may assume that $y \neq 0$ for all a $\in A$.

We have,

$$|x|^2 \leq |x + ay|^2 = [x + ay, x + ay]$$

$$= [x,x] + [x,ay] + [ay,x] + [ay,ay]$$

$$= |x|^2 + [x,y]a + a^*[y,x] + a^*|y|^2 a$$

This implies,

$$-[x,y]a - a^*[x,y] \leq a^*|y|a$$

For $a = -\frac{[y,x]}{\left\|[y,y]^{\frac{1}{2}}\right\|^2}$ the above inequality becomes

$$-[x,y]\frac{(-[y,x])}{\left\|[y,y]^{\frac{1}{2}}\right\|^2} - \frac{(-[x,y])}{\left\|[y,y]^{\frac{1}{2}}\right\|^2}[y,x] \leq \frac{-[x,y]}{\left\|[y,y]^{\frac{1}{2}}\right\|^2}|y|^2\frac{(-[y,x])}{\left\|[y,y]^{\frac{1}{2}}\right\|^2}$$

$$\frac{[x,y][y,x]}{\left\|[y,y]^{\frac{1}{2}}\right\|^2} + \frac{[x,y][y,x]}{\left\|[y,y]^{\frac{1}{2}}\right\|^2} \leq \frac{[x,y]|y|^2[y,x]}{\left\|[y,y]^{\frac{1}{2}}\right\|^4}$$

$$2\left\|[y,y]^{\frac{1}{2}}\right\|^4 [x,y][y,x] \leq \left\|[y,y]^{\frac{1}{2}}\right\|^2 [x,y]|y|^2[y,x]$$

$$2\left\|[y,y]^{\frac{1}{2}}\right\|^2 [x,y][y,x] \leq [x,y]|y|^2[y,x]$$

.

Using the inequality $a^*ba \leq \|b\| a^*a$ that holds for all $a, b \in A$, we obtain that

Comparing (1) and (2) we get

$$2\left\|[y,y]^{\frac{1}{2}}\right\|^{2}[x,y][y,x] \leq \left\|[y,y]^{\frac{1}{2}}\right\|^{2}[x,y][y,x]$$

which implies that,

$[x,y][y,x] \leq 0$. From this it follows that $[x,y]=0$. Proved.

Proposition 3.3 [4, Proposition 2.4].

Let E be a right A-module and $[.,.]$ be a C^*-semi inner product on E. Then the mapping $x \to \|[x,x]\|^{\frac{1}{2}}$ is a norm on X. Moreover, for each $x \in E$ and $a \in A$ we have $\|xa\| \leq \|x\|\|a\|$.

Theorem 3.4

Let A be a C^*-algebra and let V and W be semi-inner product A-modules. For a mapping: $T: V \longrightarrow W$ and for some $k>0$, then the following conditions are equivalent:

(i) T is A-linear and $\|Tx\| = k\|x\|$.

$[Tx,Ty] = k^2[x,y]$ for all $x,y \in X$

Proof:

(i)$\Rightarrow$(ii) for all $a \in A$,

$$\begin{aligned}\|a|Tx|\|^2 &= \|(a|Tx|)(a|Tx|)^*\| \\ &= \|a|Tx||Tx|^*a^*\| \\ &= \|a|Tx|^2a^*\| \\ &= \||a|^2|Tx|^2\| \\ &= \||T(ax)|^2\| \\ &= \|T(ax)\|^2,\end{aligned}$$

therefore since,

$$\|Tx\| = k\|x\|$$

we have,

$$\|T(ax)\|^2 = k^2\|a|x|\|^2$$

So that,

$\|a|Tx|\|^2 = k^2\|a|x|\|^2$ Taking square root on either side gives,

$\|a|Tx|\| = k\|a|x|\|$ and this yields,

$$|Tx| = |kx| \tag{3}$$

Squaring (3) we have,

$|Tx|^2 = k^2|x|^2$ thus implying that,

$$[Tx,Tx] = k^2[x,x] \tag{4}$$

Replacing x by y in the second argument of (4) we get,

$[Tx,Ty] = k^2[x,y]$.

(ii)$\Rightarrow$(i)

Given that $[Tx,Tx] = k^2[x,x] \quad \forall x,y \in X$ and $k>0$.

From the above, let $y=x$

$$[Tx,Tx] = k^2[x,x] \tag{5}$$

Taking the square root on both sides of (5) yields,

$$[Tx,Tx]^{\frac{1}{2}} = k[x,x]^{\frac{1}{2}} \tag{6}$$

Now,

$\|Tx\| = \|[Tx,Tx]\|^{\frac{1}{2}}$ by definition.

$= \left\|[Tx,Tx]^{\frac{1}{2}}\right\|$

$= \left\|k[x,x]^{\frac{1}{2}}\right\|$ by (6)

$= k\left\|[x,x]^{\frac{1}{2}}\right\|$ since $k>0$

$=k\|[x,x]\|^{\frac{1}{2}}$

$=k\|x\|$.

Theorem 3.5

Let A be a C^*-algebra and let X and Y be semi-inner product A-modules. For a mapping: $T: X \longrightarrow Y$ the following conditions are equivalent:

(i) T is orthogonal preserving mapping,
(ii) $|Tx|^2 \leq |Tx + aTy|^2$ for all $a \in A$.

Proof.

(i)$\Rightarrow$(ii)

(i) implies that $[Tx, Ty] = 0$. By the definition of Birkhoff-James it is obvious that

$|Tx|^2 \leq |Tx + aTy|^2$

$\leq [Tx, Tx + aTy] + [aTy, Tx + aTy]$

$\leq [Tx, Tx] + [Tx, aTy] + [aTy, Tx] + [aTy, aTy]$

$\leq [Tx, Tx] + [Tx, Ty]a + a^*[Ty, Tx] + [aTy, aTy]$

$\leq [Tx, Tx] + [aTy, aTy]$ since $[Tx, Ty] = 0$

Therefore,

$|Tx|^2 \leq |Tx|^2 + |ay|^2 = |Tx + aTy|$ implying (ii).

(ii)$\Rightarrow$(i)

$|Tx|^2 \leq |Tx + aTy|^2$

$\leq [Tx, Tx] + [Tx, Ty]a + a^*[Ty, Tx] + [aTy, aTy]$

Which implies that,

$-[Tx, Ty]a - a^*[Tx, Ty] \leq a^*|Ty|a$

Put $a = -\frac{[Ty,Tx]}{\||Ty|\|^2}$ the above inequality become,

$2\||Ty|\|^4[Tx, Ty][Ty, Tx] \leq \||Ty|\|^2[Tx, Ty]|Ty|^2[Ty, Tx]$

$2\||Ty|\|^2[Tx, Ty][Ty, Tx] \leq [Tx, Ty]|Ty|^2[Ty, Tx]$ (7)

$[Tx, Ty]|Ty|^2[Ty, Tx] \leq \||Ty|\|^2[Tx, Ty][Ty, Tx]$ (8)

Comparing (7) and (8) we get

$2\||Ty|\|^2[Tx, Ty][Ty, Tx] \leq \||Ty|\|^2[Tx, Ty][Ty, Tx]$

which implies that,

$[Tx, Ty][Ty, Tx] \leq 0$. From this it follows that $[Tx, Ty] = 0$.

4. Approximate Birkhoff-James Orthogonality

The approximate orthogonality in a semi-inner product space is defined by $x \perp^{\epsilon} y \Leftrightarrow |[x,y]| \leq \varepsilon\|x\|\|y\|$. However, in an arbitrary normed space X, the approximate Birkhoff-James orthogonality established by Chmielinski [10] is given by

$x \perp_B^{\varepsilon} y$ implies that $\|x + \lambda y\|^2 \geq \|x\|^2 - 2\varepsilon|\lambda|\|x\|\|y\|$, $\lambda \in \mathbb{C}$.

Motivated by the above approximate Birkhoff-James orthogonality, we present a new type of approximate orthogonality in C^*-semi-inner product space.

Proposition 4.1

For a given $\varepsilon \in [0,1)$ elements x, y in a semi-inner product A-module V are said to be approximate Birkoff-James orthogonal denoted by $x \perp_B^{\varepsilon} y$ if $\|x + ya\|^2 \geq \|x\|^2 - 2\varepsilon\|a\|\|x\|\|y\|$. ($a \in A,\ x, y \in V$).

Proof:

For any $a \in A$ we have

$\|x + ya\|^2 = \|[x + ya, x + ya]\|$

$$= \|[x,x] + [ya,ya] + [x,ya] + [ya,x]\|$$

$$\geq \|[x,x] + [ya,ya]\| - \|[x,ya] + [ya,x]\|$$

$$\geq \|[x,x]\| - \|[x,ya] + [ya,x]\|$$

$$\geq \|x\|^2 - \|[x,ya]\| - \|[ya,x]\|$$

$$\geq \|x\|^2 - 2\|a\|\|[x,y]\|$$

$$\geq \|x\|^2 - 2\varepsilon\|a\|\|x\|\|y\|$$

Thus,

$$\|x + ya\|^2 \geq \|x\|^2 - 2\varepsilon\|a\|\|x\|\|y\|$$

And so,

$x \perp_B^{\varepsilon} y$.

Corollary 4.2 Let x, y be elements in a semi-inner product A-module V such that $[x,x] \perp_B^{\varepsilon} [x,y]$, then $x \perp_B^{\varepsilon} y$.

Proof:

We assume that $x \neq 0$. Since $[x,x] \perp_B^{\varepsilon} [x,y]$

Therefore for every $a \in A$ we have,

$\|[x,x] + [x,y]a\|^2 \geq \|[x,x]\|^2 - 2\varepsilon\|a\|\|[x,x]\|\|[x,y]\|$

This implies,

$$\|[x, x + ya]\|^2 \geq \|x\|^4 - 2\varepsilon\|a\|\|x\|^2\|[x,y]\|$$

Hence we get,

$$\|x\|^2\|x + ya\|^2 \geq \|x\|^4 - 2\varepsilon\|a\|\|x\|^3\|y\|$$

Since $\|x\|^2 \neq 0$

we obtain from the above inequality,

$$\|x + ya\|^2 \geq \|x\|^2 - 2\varepsilon\|a\|\|x\|\|y\|$$

Thus, implying that,

$$x \perp_B^{\varepsilon} y$$

5. Conclusion

Approximate orthogonal preserving properties in the frame work of inner product *A*-module particularly Hilbert *C**-modules has attracted a lot of attention in recent years. In this paper we established an approximate orthogonal preserving property in the setting of *C**-semi-inner product *A*-modules. It has been observed that mappings preserving orthogonality were necessary a scalar multiple of an isometry. Further research could also be carried out on *C**-semi-inner product spaces. For instance, is it possible to define a *C**-metric on *C**-semi-inner product spaces?

Conflict of interest

The authors declare no conflict of interest.

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