Caliphate Journal of Science & Technology (CaJoST)

ISSN: 2705-313X (**PRINT**); 2705-3121 (**ONLINE**)

Research Article

Article Info

Received: 4th September 2020

Revised: 1st November 2020

Accepted: 6th November 2020

Department of Mathematical Science, Faculty of Science, Federal University Lokoja, P.M.B 1154 Kogi State Nigeria

***Corresponding author's email:**

attehmayowa48@gmail.com

Cite this: *CaJoST,* 2021, 1, 17-22

Numerical Solution of First Order Ordinary Differential Equations Using an Optimized Two-Step Block Hybrid Method

Mayowa E. Atteh* and Olaronke H. Edogbanya

A two-step block hybrid method was derived using a new set of interpolation and collocation points which were carefully chosen to improve the performance of the method. The continuous form of this method was evaluated at x_{n+2}, $x_{n+\frac{1}{2}}$ and $x_{n+\frac{5}{4}}$ the first derivative at $x_{n+\frac{5}{4}}$ to give four discrete schemes which constitute a block form. The order, consistency and convergence of the method is discussed and its accuracy established numerically.

Keywords: Block Hybrid Method, Continuous, Interpolation, Collocation, Discrete.

1. Introduction

In the field of science and engineering, mathematical models are designed to study physical phenomena such as; population growth, spread of disease, amortization of debt, rate of flow, electric circuits, satellite tracking and celestial mechanics. These models majorly resolve to differential equations either (partial or ordinary), which mostly do not have analytic solutions. Hence, the need for numerical methods as an alternative for solving them. The earliest numerical methods for solving ordinary differential equations (ODEs) are the Euler's method and Runge Kutta's method, which are single steps methods. The accuracy of the result obtained from the use of the above methods is very low and therefore cannot be used for the analysis of the model. Over the years, the desire for better performing methods has led to many research works which gave rise to the development of the linear multistep method.

Several authors such as [1], [2], [3] and [4] have done a lot of computational research on the conventional linear multistep method. The general linear multistep formula is as shown below:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \quad k \geq 2 \qquad (1)$$

The application of the above formula to solve ODEs returns results with good accuracy but poor stability. Some researchers discovered that hybridizing the linear multistep method and applying it in block form improves the accuracy, the order and the stability of the method. This is what gave birth to the block hybrid method.

[5], [7], and [6] were the first to propose the hybrid method which is a modification of the linear multistep method. In this hybrid method, the disadvantage of poor stability when the linear multistep method is applied directly is removed by lowering the step number without reducing its order. It is observed from [18] and [20] that conventional linear multistep method including the hybrid ones can be made continuous through the idea of

Multistep Collocation (MC). This approach produces piece wise polynomial solutions over k-steps $[x_n, x_{n+k}]$ for the first order ordinary differential equation. It is worth noticing that the implicit continuous multistep method interpolation is not to be directly used as the numerical integrator, but the resulting discrete multistep schemes which is derived from it, which will now be self-starting and can be applied for the solutions of initial value problems. [12] Worked on the reformation of the continuous general linear multistep method by the matrix inversion approach. In this work, she developed a method to obtain the polynomial interpolation of $\alpha_j(x)$ and $\beta_j(x)$ by inverting the D-matrix. The mathematical make-up of this matrix will be explained further in the derivation of the method. [21] Developed a continuous Butcher type two-step block hybrid method for solving first order initial value problem. The results obtained showed a class of discrete schemes of order 5 and the error constant ranging from C_6= 1.45 x 10^{-5} to C_6 = 1.790 x 10^{-4}. [11] Went further to increase the step size and introduce the Butcher type three-step block hybrid method. They employed the multistep collocation approach, and this yielded a class of 2 discrete schemes of order 7 with error constant $C_8 = \frac{27}{777420}$ and $C_8 = \frac{155525}{1273724928}$. This shows that an increment in the step size will increase the accuracy of the method but may have adverse effect on the stability.

In this present consideration, we seek to improve the accuracy of this method by interpolating and collocating (1) for various set of interpolation and collocation points. The best performing set was adopted and presented in this paper.

1.1 Definition: Order and Error constant

A linear multistep method is of order p, if $c_0 = c_1 = c_2 \ldots c_p = 0$ and $c_{p+1} \neq 0$. c_{p+1} Is called the error constant of the method.

where

$$c_0 = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k$$

$$c_1 = (\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k) - (\beta_0 + \beta_1 + \cdots + \beta_k)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$c_q = \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + \cdots + k^q\alpha_k) - \frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2 + \cdots + k^{q-1}\beta_k)$$

1.2 Definition: Consistency

A linear multistep method is consistent, if it has order $p \geq 1$

1.3 Definition: Zero-stability

A block method is zero stable provided $\lambda_j, j = 1(1)s$ of the first characteristic polynomial p(λ) specified as $p(\lambda) = det\left|\sum_{i=0}^{s} A^{(1)} \lambda^{(s-1)}\right| = 0$ satisfies $|\lambda_j| \leq 1$ and for those roots with $|\lambda_j| = 1$, the multiplicity must not exceed two. The principal root of $p(\lambda)$ is denoted by $\lambda_1 = \lambda_2 = 1$

1.4 Definition: Convergence

A linear multistep method is said to be convergent, if the numerical solution approaches the exact solution as the step size h tends to zero. The necessary and sufficient condition for a linear multistep method to be convergent is that it must be consistent and zero-stable [14].

1.5 Definition: A-Stable

A numerical method is said to be A-stable, if its region of absolute stability contains the whole of the left-hand half plane $Re\ \lambda < 0$ [8].

1.6 Definition: A(α)-Stable

A numerical method is said to be A(α)-stable, $\alpha \in (0, \frac{\pi}{2})$, if its region of absolute stability contains the infinite wedge $W_\alpha = [h\lambda\ -\alpha < \pi - argh\ \lambda]$, it is said to be A(0)-stable if it is A(α)-stable for some(sufficiently small) $\alpha \in (0, \frac{\pi}{2})$ [9].

1.7 Definition: Absolute stability

A numerical method is said to be absolutely stable for a given $h\lambda$, if all the roots of $\pi(0, h\lambda)$ lie within a unit circle. A region R_A of the complex plane is said to be a region of absolute stability if the method is stable for all $h\lambda$ in R_A.

2. Derivation of the Proposed Method

To develop the continuous scheme and its discrete scheme we use the approach introduced in [12] where a k-step multistep collocation method was obtained as follows;

$$\bar{y}(x) = \sum_{j=0}^{t-1} \alpha_j(x) y(x_{n+j}) + h \sum_{j=0}^{m-1} \beta_j(x) f(\bar{x}_j, y(\bar{x}_j)) \quad (2.1)$$

where the polynomial interpolation of the continuous coefficients $\alpha_j(x)$ and $\beta_j(x)$ is expressed below.

$$(x) = \sum_{i=0}^{t+m-1} \alpha_{j,i+1}\, x^i \qquad (2.2)$$

$$h\beta_j(x) = \sum_{i=0}^{t+m-1} h\beta_{j,i+1}\, x^i \qquad (2.3)$$

x_{n+j} for $j = 0, 1, \ldots, t-1$ in (3.1) are t arbitrary chosen interpolation points from $\{x_n, \ldots, x_{n+k}\}$ and $\bar{x}_j$ for $j = 0, 1, \ldots, m-2$ are the m collocation points belonging to $\{x_n, \ldots, x_{n+k}\}$.

Equation (2.2) and (2.3) give rise to a matrix equation of the form $DC = 1$ [21].

Where

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m+1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2\bar{x}_0 & \cdots & (t+m-1)\bar{x}_0^{t+m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2\bar{x}_{m-1} & \cdots & (t+m-1)\bar{x}_{m-1}^{t+m-2} \end{bmatrix} \qquad (2.4)$$

And similarly, equation (2.4) and (2.5) becomes (2.7) and (2.8) respectively;

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x^2_{n+\frac{1}{2}} & 4x^3_{n+\frac{1}{2}} & 5x^4_{n+\frac{1}{2}} \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 \end{bmatrix} \qquad (2.7)$$

and

$$C = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{t-1,1} & h\beta_{0,1} & \cdots & h\beta_{m-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{t-1,2} & h\beta_{0,2} & \cdots & h\beta_{m-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,t+m} & \alpha_{1,t+m} & \cdots & \alpha_{t-1,t+1} & h\beta_{0,t+m} & \cdots & h\beta_{m-1,t+m} \end{bmatrix} \qquad (2.5)$$

and I is the $6x6$ identity matrix.

when $k = t = 2$ and $m = 4$ then equation (2.1) becomes.

$$\bar{y}(x) = \alpha_0 y_n + \alpha_1 y_{n+1} + h\left[\beta_0 f_n + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_1 f_{n+1} + \beta_2 f_{n+2}\right] \qquad (2.6)$$

Note:

$\beta_0(x) = \beta_0, y(x_{n+1}) = y_{n+1}, f(\overline{x_n}, y(\overline{x_n})) = f_n$

$$\bar{y}'(x) = \left[\frac{120(x-x_n)^4 - 420(x-x_n)^3h + 420(x-x_n)^2h^2 - 120(x-x_n)h^3}{h^5}\right]y_n$$
$$+\left[\frac{-120(x-x_n)^4 + 420(x-x_n)^3h - 420(x-x_n)^2h^2 + 120(x-x_n)h^3}{h^5}\right]y_{n+1}$$
$$+\left[\frac{80(x-x_n)^4 - 284(x-x_n)^3h + 294(x-x_n)^2h^2 - 94(x-x_n)h^3 + 4h^4}{4h^4}\right]f_n$$
$$+\left[\frac{240(x-x_n)^4 - 832(x-x_n)^3h + 816(x-x_n)^2h^2 - 224(x-x_n)h^3 +}{3h^4}\right]f_{n+\frac{1}{2}}$$
$$+\left[\frac{20(x-x_n)^4 - 72(x-x_n)^3h + 75(x-x_n)^2h^2 - 22(x-x_n)h^3 +}{h^4}\right]f_{n+1}$$
$$+\left[\frac{4(x-x_n)^3 - 6(x-x_n)^2h + 2(x-x_n)h^2}{12h^3}\right]f_{n+2} \qquad (2.10)$$

$$C = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & h\beta_{0,1} & h\beta_{1,1} & h\beta_{2,1} & h\beta_{3,1} \\ \alpha_{0,2} & \alpha_{1,2} & h\beta_{0,2} & h\beta_{1,2} & h\beta_{2,2} & h\beta_{3,2} \\ \alpha_{0,3} & \alpha_{1,3} & h\beta_{0,3} & h\beta_{1,3} & h\beta_{2,3} & h\beta_{3,3} \\ \alpha_{0,4} & \alpha_{1,4} & h\beta_{0,4} & h\beta_{1,4} & h\beta_{2,4} & h\beta_{3,4} \\ \alpha_{0,5} & \alpha_{1,5} & h\beta_{0,5} & h\beta_{1,5} & h\beta_{2,5} & h\beta_{3,5} \\ \alpha_{0,6} & \alpha_{1,6} & h\beta_{0,6} & h\beta_{1,6} & h\beta_{2,6} & h\beta_{3,6} \end{bmatrix} \qquad (2.8)$$

The columns of the matrix $C = D^{-1}$ consist of the coefficients of $\alpha_j(x)$ and $\beta_j(x)$. To obtain the elements of C we have to invert the matrix D using the mathematical application Maple. After the inversion and appropriate substitutions, we obtain the desired two-step continuous hybrid scheme as shown.

$$\bar{y}(x) = \left[\frac{24(x-x_n)^5 - 105(x-x_n)^4h + 140(x-x_n)^3h^2 - 60(x-x_n)^2h^3 + h^5}{h^5}\right]y_n$$
$$+\left[\frac{-24(x-x_n)^5 + 105(x-x_n)^4h - 140(x-x_n)^3h^2 + 60(x-x_n)^2h^3}{h^5}\right]y_{n+1}$$
$$+\left[\frac{16(x-x_n)^5 - 71(x-x_n)^4h + 98(x-x_n)^3h^2 - 47(x-x_n)^2h^3 + 4(x-x_n)h^4}{4h^4}\right]f_n$$
$$+\left[\frac{48(x-x_n)^5 - 208(x-x_n)^4h + 272(x-x_n)^3h^2 - 112(x-x_n)^2h^3 +}{3h^4}\right]f_{n+\frac{1}{2}}$$
$$+\left[\frac{4(x-x_n)^5 - 18(x-x_n)^4h + 25(x-x_n)^3h^2 - 11(x-x_n)^2h^3 +}{h^4}\right]f_{n+1}$$
$$+\left[\frac{(x-x_n)^4 - 2(x-x_n)^3h + (x-x_n)^2h^2}{12h^3}\right]f_{n+2} \qquad (2.9)$$

Differentiating (2.9) with respect to$x$, we have.

On evaluating (2.9) at $x = x_{n+2}, x_{n+\frac{1}{2}}, x_{n+\frac{5}{4}}$ and (2.10) at$x_{n+\frac{5}{4}}$, we obtain the following four discrete schemes which will be applied in block form:

$$y_{n+2} - 32y_{n+1} + 31y_n = \frac{h}{3}\left[f_{n+2} - 15f_n - 64f_{n+\frac{1}{2}} - 12f_{n+1}\right] \qquad (2.11)$$

$$y_{n+\frac{1}{2}} - \frac{53}{16}y_{n+1} + \frac{37}{16}y_n = \frac{h}{192}\left[f_{n+2} - 69f_n - 352f_{n+\frac{1}{2}} - 120f_{n+1}\right] \qquad (2.12)$$

$$y_{n+\frac{5}{4}} - \frac{875}{256}y_{n+1} + \frac{619}{256}y_n = \frac{h}{3072}\left[25f_{n+2} - 1185f_n - 5200f_{n+\frac{1}{2}} - 300f_{n+1}\right] \qquad (2.13)$$

$$\frac{675}{32}y_n - \frac{675}{32}y_{n+1} = \frac{h}{192}\left[15f_{n+2} - 648f_n - 2820f_{n+\frac{1}{2}} - 405f_{n+1} - 192f_{n+\frac{5}{4}}\right] \quad (2.14)$$

Equations (2.11), (2.12), (2.13) and (2.14) constitutes the members of a zero-stable block integrator. The application of the block integrator with $n = 0$ gives the values of y_1 and y_2 as shown in table (1-3).

3. Results and Discussion

To evaluate the performance of the derived method, we need to consider some of its important properties. These properties include the order of the method, the error constant, zero-stability, the consistency and the convergence.

3.1 Order and Error constant

Using the formula stated in definition (1.1) the order of the derived 1-block 4-point method is $(5,5,5,5)^T$ and the error constant is $C_6 = C_{p+1} = (\frac{17}{180}, -\frac{13}{46080}, -\frac{215}{589824}, -\frac{27}{8192})$.

3.2 Consistency

Since the order of the block integrator is greater than one, then by definition (1.2) it is consistent.

3.3 Zero-stability

The first characteristics polynomial of the 1-block 4-point method is:

$$P(R) = det\left|RA^{(0)} - A^{(1)}\right|$$

$$= det\left[R\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} - \begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}\right]$$

$$= det\begin{bmatrix}R&0&0&-1\\0&R&0&-1\\0&0&R&-1\\0&0&0&R-1\end{bmatrix}$$

$$= [R^3(R-1)]$$

$$\Rightarrow R_1 = R_2 = R_3 = 0 \quad or \quad R_4 = 1$$

From the above computations and following definition (1.3), the 1-block 4-point method is zero- stable.

3.4 Convergence

Since the derived 1-block 4-point method is consistent and also zero-stable, it is therefore convergent following definition (1.4).

4. Numerical Experiment

Example 1

$y' = -y, y(0) = 1, h = 0.1, 0 \le x \le 1$, exact analytic solution $y(x) = e^{-x}$

Example 2

From [4], consider the differential equation of growth model of the form:

$y' = \alpha y, y(0) = 1000, t \in [0,1], h = 0.1$

The equation above represents the rate of growth of bacteria in a colony. We shall assume the model grows continuously and without restriction. One may ask how many bacteria are in the colony after some hours if an individual produce an average of 0.2 offspring every hour?

We assume that $y(t)$ is the population size at time t. This therefore implies that the equation maybe written as:

$y' = 0.2y, y(0) = 1000, t \in [0,1], h = 0.1$

Method with the exact solution, [4] and [21].

Table 1. Comparison of the result of our block hybrid method with exact solution and [21]

x	Exact solution	Present work	[21]
0	1.0000000000	1.0000000000	1.0000000000
0.1	0.9048374180	0.9048374179	0.9048374190
0.2	0.8187307531	0.8187307536	0.8187307547
0.3	0.7408182207	0.7408182210	0.7408182186
0.4	0.670320046	0.6703200464	0.6703200387
0.5	0.6065306597	0.6065306600	0.6065306501
0.6	0.5488116361	0.5488116360	0.5488116201
0.7	0.4965853038	0.4965853037	0.4965852654
0.8	0.4493289641	0.4493289672	0.4493288879
0.9	0.4065696597	0.4065696625	0.4065695729
1	0.3678794412	0.3678794429	0.3678793518

Table 2. Absolute error for Example 1

x	Error of present work	[21]
0	0.0000000000	0.0000000000
0.1	1×10^{-10}	1×10^{-9}
0.2	5×10^{-10}	1.6×10^{-9}
0.3	3×10^{-10}	2.1×10^{-9}
0.4	4×10^{-10}	7.3×10^{-9}
0.5	3×10^{-10}	9.6×10^{-9}
0.6	1×10^{-10}	1.6×10^{-8}
0.7	1×10^{-10}	3.84×10^{-8}
0.8	3.1×10^{-9}	7.62×10^{-8}
0.9	2.8×10^{-9}	8.68×10^{-8}
1	1.7×10^{-9}	8.94×10^{-8}

Table 3. Comparison of the result of our block

x	Exact solution	Present work	[21]	[4]
0	1000.000000	1000.000000	1000.000000	1000.000000
0.1	1020.201340	1020.201340	1020.201340	1020.20135498
0.2	1040.810774	1040.810774	1040.810774	1040.81079102
0.3	1061.836547	1061.836546	1061.83654	1061.83654785
0.4	1083.287068	1083.287066	1083.287068	1083.28710938
0.5	1105.170918	1105.170916	1105.170918	1105.17102051
0.6	1127.496852	1127.496838	1127.496852	1127.49694824
0.7	1150.273799	1150.273832	1150.273798	1150.27392578
0.8	1173.510871	1173.510891	1173.510823	1173.51098633
0.9	1197.217363	1197.217384	1197.217312	1197.21752930
1	1221.402758	1221.402792	1221.402692	1221.40295410

Table 4. Absolute error for Example 2

x	Error of present work	[21]	[4]
0	0.000000000	0.000000000	0.000000000
0.1	0.000000000	0.000000000	1.498×10^{-5}
0.2	0.000000000	0.000000000	1.702×10^{-5}
0.3	1×10^{-6}	7×10^{-6}	8.5×10^{-7}
0.4	2×10^{-6}	0.000000000	4.138×10^{-5}
0.5	2×10^{-6}	0.000000000	1.0251×10^{-4}
0.6	1.4×10^{-5}	0.000000000	9.624×10^{-5}
0.7	3.3×10^{-5}	1×10^{-6}	1.2678×10^{-4}
0.8	2×10^{-5}	4.8×10^{-5}	1.1533×10^{-4}
0.9	2.1×10^{-5}	5.1×10^{-5}	1.663×10^{-4}
1	3.4×10^{-5}	6.6×10^{-5}	1.961×10^{-4}

4. Conclusion

We derived an optimized hybrid block method. The performance of the block hybrid numerical integrator was optimized by:

(a) Testing for various evaluation points.

(b) Checking for best performing interpolation and collocation points.

Also, the numerical integrator possesses the following desirable qualities:

(a) The order is of the integrator is 5, which is considerably high.

(b) It is consistent.

(c) It is zero-stable and convergent.

Finally, the derived numerical integrator was used to solve examples and the results obtained compares favorably with the exact solution and result from other cited works.

Conflict of interest

The authors declare no conflict of interest.

References

[1] Lambert J. D. *Computational methods in ordinary differential equations. John Wiley and Sons*. New York, 1973.

[2] Onumanyi P., Awoyemi D. O., Jator S. N. and Sirisena U. W. "New linear multistep methods with continuous coefficients for first order initial value problems*". Journal of the Nigerian Mathematical Society,* 1994.

[3] Fatunla S. O. *Numerical methods for initial value problems in ordinary differential equations*.New York, Academic press, 1988.

[4] Sunday J. and Odekunle, M. R. "A new numerical integrator for the solutions of initial value problems in ordinary differential Equations". *Pacific Journal of Science and Technology.* 13(1), p. 215-221, 2012.

[5] Gragg W. B. and Stetter H. J. Generalized multistep predictor corrector methods. *Journal of the Association for Computing Machinery*. 11. p.188-209, 1964.

[6] Gear C. W. *Numerical IVPs in ODEs*. Prentice Hall. Englewood Cliffs, N. I., 1971.

[7] Butcher J. C. (1965). "A modified multistep method for the numerical integration of ordinary differential equation." *Journal of Association Computational. Mathematics.* 12, p.124-135, 1965.

[8] Dahlquist G. "A special stability problem for linear multistep methods." *BIT Numerical Mathematics*, 3, p.27-43. 1963.

[9] Widlund O. B. "Notes on unconditional stable linear multistep methods." *BIT Numerical Mathematics* 7, p. 65-70, 1967.

[10] Sirisena U. W., Kumleng G. M. and Yahaya Y. A. "A new butcher type two-step block hybrid multistep method for accurate and efficient parallel solution of ODEs." *Abacus Maths Series*, 31, p. 1-7, 2004.

[11] Areo E. A., Ademiluyi R. A. and Babatola, P. O. "Accurate collocation multistep method for integration of first order ordinary differential equations." *Journal of Mordern Mathematics and Statistics,* 2, p. 1-6, 2008.

[12] Sirsena U. W. "A reformation of the continuous general linear multistep method by matrix inversion for first order initial value problems." Ph.D. thesis (Unpublished). University of Ilorin: Ilorin, Nigeria. 1997.

[13] Awari Y. S., Abada, A. A., Emma P. M. and Kamoh N. M. "Application of two-step continous hybrid Butcher's method in block form for the solution of first order initial value problem". *Academic Research International*. p. 209-218. (2013).

[14] Henrici P. *Discrete variable methods for ODE's*. New York: John Wiley, 1962.

[15] Lambert J. D. *Numerical methods for ordinary differential Systems: the initial value problem*. John Willey and Sons, New York, 1991.

[16] Awoyemi, D. O. "On some continuous linear multistep methods for initial value problem." unpublished Phd thesis, University of Ilorin, Ilorin Nigeria, 1992.

[17] Sirisena U. W. and Onumanyi P. "A modified continuous numerical method for second order ODE's." *Nigerian Journal of Mathematics and Applications*.7, p.123-129,1994.

[18] Lie I. and Norsett S. P. "Super convergence for multistep collocation." *Mathematics of Computation*. 52, p.65-79, 1989.

[19] Areo E. A. and Adeniyi R. B. "A self-starting linear multistep method for direct solution of initial value problems for second order ordinary differential equations." *International Journal of Pure and Applied Mathematics* 82(3), p.345-364, 2013.

[20] Onumanyi P., Sirisena U. W. and Jator S. N. "Continuous finite difference approximations for solving differential equations." *International Journal of Computer Mathematics,* 1999.

[21] Sagir A. M. "Numerical treatment of block Methods for the solution of ordinary differential equations." *International. Journal Mathem*.

