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<sup>1</sup>Department of Mathematics, Sokoto State University, Sokoto State, Nigeria.

<sup>2</sup>Department of Mathematics, Abdullahi Fodio University of Science and Technology Aleiro, Kebbi State, Nigeria.

<sup>3</sup>Department of Mathematics, Usman Dan Fodio University, Sokoto State, Nigeria.

\*Corresponding author's email:

[Buharimamuda2018@gmail.com](mailto:Buharimamuda2018@gmail.com)

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## Weak Convergence Theorem for Attractive Point of Finitely Many Families Generalized Nonexpansive Mappings

Buhari Mamuda<sup>1</sup>, Sani F. Abubakar<sup>2</sup>, Abor I. Garba<sup>3</sup>

In this paper, we study iterative approximation of attractive points for finitely many families of generalized nonexpansive mappings in a uniformly convex Banach space. We introduce a new algorithm combining a viscosity step with an inertial extrapolation. Under suitable control of the inertial and viscosity parameters and standard conditions on the mappings, we prove that the generated sequence is bounded. We then show that every weak cluster point of the sequence is an attractive point common to all mapping families. The main result establishes that the sequence converges weakly to a unique such attractive point. This extends earlier results confined to two mappings by considering finite family. These findings confirm that the proposed viscosity-inertial iteration successfully approximates the common attractive point under the stated hypotheses. Overall, the work broadens convergence theory in Banach spaces by enabling new classes of algorithms for approximating solution points of generalized nonlinear problems.

**Keywords:** Weak convergence; viscosity; attractive point; uniformly convex Banach space; generalized nonexpansive mapping.

## 1. Introduction

Let  $X$  be a real Banach space and  $C$  a nonempty closed and convex subset of  $X$ , Also  $T$  be a mapping from  $C$  to  $C$ , the set of fixed point of  $T$  is denoted by

$$F(T) = \{x \in C: Tx = x\}.$$

A mapping  $T: C \rightarrow C$  is called

1. non-expansive mapping if  $\|Tx - Ty\| \leq \|x - y\| \forall x, y \in C$ .

2. Quasi-nonexpansive mapping if  $F(T) \neq \emptyset$ , and

$$\|Tx - z\| \leq \|x - z\| \forall x \in C \text{ and every } z \in F(T).$$

3. generalized nonexpansive mapping, if

$$\|Tx - Ty\| \leq a \|x - y\| + b \|Tx - x\| + c \|Ty - y\| + d \|Tx - x\| + e \|Ty - y\|$$

$\forall x, y \in C, a, b, c, d, e$ , are non-negative constant, and  $a + b + c + d + e \leq 1$ .

It was also mentioned that the above definition is equivalent with

$$\|Tx - Ty\| \leq a \|x - y\| + b (\|Tx - x\| + \|Ty - y\|) + c (\|Tx - y\| + \|Ty - x\|)$$

$a + 2b + 2c \leq 1$  if  $b$  and  $c = 0$ , we have

$$\|Tx - Ty\| \leq a \|x - y\|.$$

A point is termed an attractive point if it meets the following condition:

$$\|Tx - u\| \leq \|x - u\| \forall x \in C, \text{ and } u \in X$$

Numerous researchers have explored attractive points from various perspectives. Takahashi and Takeuchi (2011) were pioneers in introducing the concept of attractive points within Hilbert space. Their primary aim was to eliminate the reverse closure and closedness assumptions typically applied to a nonempty subset in Ballon's renowned nonlinear ergodic theorem (Ballon's, 1975). They also established an existence theorem for attractive points without requiring convexity. It is clear from the definition of fixed point sets that, in general, a fixed point does not necessarily qualify as an attractive point, and vice versa. Jenwit and Suthep (2020) presented an accelerated iterative algorithm for computing a common fixed a common fixed point of an infinite family of nonexpansive mapping in Hilbert space with a structure similar to Nesterov's(1983) acceleration principal approach which laid the foundation for modern inertial methods in optimization and fixed point.

Ali and Ali (2019) demonstrated a convergence theorem for the Mann iteration applied to two

generalized nonexpansive mappings in uniformly convex Banach space.

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n x_n + \beta_n T_1 x_n + \gamma_n T_2 x_n \forall n \in \mathbb{N} \end{cases} \quad (1.1)$$

Where  $\alpha_n, \beta_n, \gamma_n \in (0, 1)$  and  $\alpha_n + \beta_n + \gamma_n = 1$ .

Thongpaen, Kaewkhao, Phudolsitthiphath, Suantai, and Inthakon (2021). Established weak and strong convergence of (1.1) for common attractive points of widely more generalized hybrid mappings in Hilbert space.

Thongphaen, Warunun, Suthep, and Narawadee (2022) demonstrated strong convergence results for common attractive of two generalized non-expansive mappings of (1.1) in uniformly convex Banach space using the condition of a non-decreasing function with and for any such that that is  $\exists$  a non-decreasing continuous function  $h: [0, \infty) \rightarrow [0, \infty)$  with and  $h(0) = 0$  and  $h(r) > 0$  for any  $r > 0$  such that

$$\begin{aligned} h(d(x_n, A(T_1, T_2))) &\leq \|x_n - T_1 x_n\| \text{ or} \\ h(d(x_n, A(T_1, T_2))) &\leq \|x_n - T_2 x_n\|. \end{aligned}$$

Wiriapongsanon, Inthakon and Phudolstthiphath (2022) studied finite families of multivalued nonexpansive mappings for common attractive and proved convergence of modified iterative schemes in uniformly convex settings. Read also (Ali, Haruna and Harbau, 2023; see also Asi, Khan, and Khan, 2023).

A new inertial accelerated Mann’s algorithm was also introduced for finding a point in the set of fixed points of asymptotically non-expansive mapping in a real uniformly convex Banach space, which was able to established weak and strong convergence theorems for the scheme (Harbau, Godwin, Lateef, and Ahmad , 2021).

In a related development Harbau, Godwin, and Malik (2024) proposed a viscosity approximation method and proved a strong convergence theorem for the attractive point of a finite family of widely more generalized hybrid mappings.

Recent advances in iterative methods for generalized nonexpansive mappings in Banach spaces have focused on developing robust schemes with strong convergence properties under minimal conditions. Strong convergence theorems for multivalued generalized nonexpansive maps using modified F-iteration processes in uniformly convex Banach spaces have also been established (Hussain, Alamri and Alsulami, 2023). Further, Suzuki-type generalized nonexpansive mappings and associated convergence analysis highlight the

broader class of mappings where iterative approximation remains effective (Ekinci & Temir, 2023). Finally, inertial iterative methods that combine inertial effects with fixed point and variational inequality frameworks in uniformly convex Banach spaces offer insight into accelerating convergence behaviors relevant to your viscosity–inertia approach (Aldosary & Farid, 2024). Ivanov (2025) investigated uniformly convex sets in banach spaces, extending classical geometric notions from normed spaces and analyzing several variants of uniform convexity along their structural properties. Stability of functional equations in uniformly convex settings has been established (Sun & Zhang, 2023).

Building upon the results of Thongphaen et al. (2022) as well as the abovementioned results, the aim of this research aims to investigate a viscosity-Inertia approximation method for finitely Many families generalized nonexpansive mappings. We establish weak convergence theorems to attractive point of these mappings.

## 2. Preliminaries

In this section we give some basic definitions of some terms and lemmas that will be used in our main work

**Definition 2.1:** A Banach space  $X$  is called uniformly convex space if for any  $\epsilon \in (0, 2]$  there exists a  $\sigma = \sigma(\epsilon)$  such that  $x, y \in X$  with  $\|x\|, \|y\| \leq 1$  and  $\|x - y\| \geq \epsilon$ , then  $\|\frac{x+y}{2}\| \leq 1 - \sigma$ .

**Definition 2.2:**

1. Demiclosed at  $y_0 \in C$ . If for any  $\{x_n\} \in C$  which converges weakly to  $x_0 \in C$  and  $Tx_0 = y_0$ , it holds that  $Tx_0 = y_0$ .
2. Semicompact, if for any  $\{x_n\} \in C$  such that  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ . There exists subsequence  $x_{n_k}$  from  $x_n$  such that  $x_{n_k} \rightarrow x \in C$ .

Remark 2.1: Every uniformly convex Banach space is Reflexive.

**Lemma 2.1 [Goebel and Kirk (1990)]:** Suppose that  $T: H \rightarrow H$  is a non-expansive mapping. Let  $\{x_n\}_{n=1}^{\infty}$  be a vector in  $H$ , and let  $u$  be a vector in  $H$ . If  $x_n \rightarrow u$  and  $x_n - Tx_n \rightarrow 0$  as  $n \rightarrow \infty$  then  $u \in A(T)$ .

**Lemma 2.2 [Opial (1967)]:** if a Banach space  $X$  having a weakly continuous duality mapping  $J$  and  $x_n \rightarrow x_0$  then for any  $x \in X$   
 $\liminf_{n \rightarrow \infty} \|x_n - x\| > \liminf_{n \rightarrow \infty} \|x_n - x_0\|$ .

**Lemma 2.3 [Yeol, Haiyun and Ginti (2002)]:** Let  $X$  be a uniformly convex Banach space and  $B_r(0)$  and

$r > 0$  be a closed ball of  $X$ . There exists a  $g: [0, \infty) \rightarrow [0, \infty)$  with  $g(0) = 0$  such that continuous strictly increasing function  $\|\alpha x + \beta y + \lambda z\|^2 \leq \alpha \|x\|^2 + \beta \|y\|^2 + \lambda \|z\|^2 - \alpha\beta g(\|x - y\|)$  for all  $x, y \in B_r(0)$  and  $\alpha, \beta, \lambda \in (0, 1)$  with  $\alpha + \beta + \lambda = 1$

**Lemma 2.4 [ Zhang (2009)]:** Let  $X$  be a uniformly convex Banach space and  $B_r(0)$  and  $r > 0$ . Let  $B_r(0) = \{x \in X: \|x\| \leq r\}$ . Then for any  $x_n \subset B_r(0)$ , for any  $\lambda_j \in (0, 1), j \in \{1, 2, 3, \dots, s\}$  with  $\sum_{j=1}^s \lambda_j = 1$ , there exists a continuous strictly increasing function  $g: [0, \infty) \rightarrow [0, \infty)$  with  $g(0) = 0$  such that for any  $j, i, k \in \{1, 2, 3, \dots, s\}$  with  $j < i, k$  the following inequality holds:

$$\left\| \sum_{j=1}^s \lambda_j x_j \right\|^2 \leq \sum_{j=1}^s \lambda_j \|x_j\|^2 - \lambda_i \lambda_k g(\|x_j - x_k\|).$$

### 3. Results

In this section we prove strong convergence

Theorem 3.1:

Let  $X$  be a uniformly convex Banach space and  $C$  a nonempty convex subset of  $X$  satisfying opial's condition. Suppose  $T_i: C \rightarrow C$  for every  $i \in \{1, 2, \dots, K\}$  be a finitely many families generalized nonexpansive mapping with  $\bigcap_{i=1}^K A(T_i) \neq \emptyset$ . Let  $\{u_n\}$  be a sequence generated by

$$\left\{ \begin{array}{l} u_0, u_1 \in C \\ v_n = u_n + \varepsilon_n(u_n - u_{n-1}), \quad \forall n \geq 1 \\ w_n = \sum_{i=1}^{K+1} \gamma_{n,i} T_{i-1} v_n \\ u_{n+1} = \alpha_n f(u_n) + \beta_n u_n + \omega_n w_n \quad \forall n \geq 1 \end{array} \right. \quad (3.0)$$

Where  $f: C \rightarrow C$  is a contraction mapping with  $L \in [0, 1)$ ,  $T_0$  is identity mapping. The sequences  $\alpha_n, \beta_n, \omega_n$  and  $\gamma_{n,i} \in (0, 1)$  and  $\sum_{i=1}^{K+1} \gamma_{n,i} = 1 \forall n \geq 1$  and the following conditions holds:

1.  $\alpha_n + \beta_n + \omega_n = 1 \forall n \geq 1$ ,
2.  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
3.  $\lim_{n \rightarrow \infty} \varepsilon_n \|u_n - u_{n-1}\| = 0$  and  $\lim_{n \rightarrow \infty} \frac{\varepsilon_n}{\alpha_n} \|u_n - u_{n-1}\| = 0$ ,
4.  $0 < a \leq \beta_n \leq b < 1$ ,
5.  $0 < c \leq \omega_n \leq d < 1$ .

Then  $\{u_n\}$  converges weakly to  $x \in \bigcap_{i=1}^K A(T_i)$ .

Proof:

We first show that  $\{u_n\}$  is bounded. By the definition of  $\{v_n\}$  and  $\{w_n\}$ , we have

$$\begin{aligned} \|v_n - x\| &= \|u_n + \varepsilon_n(u_n - u_{n-1}) - x\| \\ \|v_n - x\| &\leq \|u_n - x\| + \varepsilon_n \|u_n - u_{n-1}\|. \end{aligned} \quad (3.1)$$

Also, we have

$$\begin{aligned}
\|w_n - x\| &= \left\| \sum_{i=1}^{K+1} \gamma_{n,i} T_{i-1} v_n - x \right\| \\
&= \left\| \sum_{i=1}^{K+1} \gamma_{n,i} (T_{i-1} v_n - x) \right\| \\
&\leq \sum_{i=1}^{K+1} \gamma_{n,i} \|T_{i-1} v_n - x\| \\
&\leq \sum_{i=1}^{K+1} \gamma_{n,i} \|v_n - x\| \\
&\leq \|v_n - x\| \\
\|w_n - x\| &\leq \|u_n - x\| + \varepsilon_n \|u_n - u_{n-1}\| \tag{3.2}
\end{aligned}$$

Now using equations(3.1) and (3.2), we have

$$\begin{aligned}
\|u_{n+1} - x\| &= \|\alpha_n f(u_n) + \beta_n u_n + \omega_n w_n - x\| \\
&= \|\alpha_n (f(u_n) - x) + \beta_n (u_n - x) + \omega_n (w_n - x)\| \\
&= \|\alpha_n (f(u_n) - f(x)) + (f(x) - x) + \beta_n (u_n - x) + \omega_n (w_n - x)\| \\
&\leq \alpha_n \|f(u_n) - f(x)\| + \alpha_n \|f(x) - x\| + \beta_n \|u_n - x\| + \omega_n (\|u_n - x\| + \varepsilon_n \|u_n - u_{n-1}\|) \\
&\leq \alpha_n L \|u_n - x\| + \beta_n \|u_n - x\| + \omega_n \|u_n - x\| + \omega_n \varepsilon_n \|u_n - u_{n-1}\| + \alpha_n \|f(x) - x\| \\
&= \alpha_n L \|u_n - x\| + (1 - \alpha_n) \|u_n - x\| + \omega_n \varepsilon_n \|u_n - u_{n-1}\| + \alpha_n \|f(x) - x\| \\
&= (\alpha_n L + 1 - \alpha_n) \|u_n - x\| + \omega_n \varepsilon_n \|u_n - u_{n-1}\| + \alpha_n \|f(x) - x\| \\
&= [1 - \alpha_n(1 - L)] \|u_n - x\| + \omega_n \varepsilon_n \|u_n - u_{n-1}\| + \alpha_n \|f(x) - x\| \\
&= [1 - \alpha_n(1 - L)] \|u_n - x\| + \frac{\alpha_n(1-L)}{\alpha_n(1-L)} [\omega_n \varepsilon_n \|u_n - u_{n-1}\| + \alpha_n \|f(x) - x\|] \\
&= [1 - \alpha_n(1 - L)] \|u_n - x\| + \alpha_n(1 - L) \left[ \frac{\omega_n}{1-L} \cdot \frac{\varepsilon_n}{\alpha_n} \|u_n - u_{n-1}\| + \frac{1}{1-L} \|f(x) - x\| \right].
\end{aligned}$$

By condition 3, we have  $\lim_{n \rightarrow \infty} \frac{\varepsilon_n}{\alpha_n} \|u_n - u_{n-1}\| = 0$ ,  $\exists P_1 > 0$  such that  $\frac{\varepsilon_n}{\alpha_n} \|u_n - u_{n-1}\| \leq P_1$

$\forall n \geq 1$ ,

$$\begin{aligned}
\|u_{n+1} - x\| &\leq [1 - \alpha_n(1 - L)] \max \left\{ \|u_n - x\|, \frac{dP_1}{1-L} + \frac{\|f(x) - x\|}{1-L} \right\} \\
&\quad + \alpha_n(1 - L) \max \left\{ \|u_n - x\|, \frac{dP_1}{1-L} + \frac{\|f(x) - x\|}{1-L} \right\} \\
&\leq \max \left\{ \|u_n - x\|, \frac{dP_1}{1-L} + \frac{\|f(x) - x\|}{1-L} \right\} \\
&\quad - \alpha_n(1 - L) \max \left\{ \|u_n - x\|, \frac{dP_1}{1-L} + \frac{\|f(x) - x\|}{1-L} \right\} \\
&\quad + \alpha_n(1 - L) \max \left\{ \|u_n - x\|, \frac{dP_1}{1-L} + \frac{\|f(x) - x\|}{1-L} \right\} \\
\|u_{n+1} - x\| &\leq \max \left\{ \|u_n - x\|, \frac{dP_1}{1-L} + \frac{\|f(x) - x\|}{1-L} \right\}.
\end{aligned}$$

By induction, we have

$$\|u_n - x\| \leq \max \left\{ \|u_0 - x\|, \frac{dP_1}{1-L} + \frac{\|f(x) - x\|}{1-L} \right\} \quad \forall n \geq 1. \tag{3.3}$$

Since  $\|u_n - x\|$  is bounded, this imply  $\{u_n\}$  is bounded. Hence  $\{v_n\}, \{w_n\}, \{f(u_n)\}, \{T_i u_n\}, \{T_i v_n\}$  are all bounded.

Next, we show  $u_n - T_i u_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Before, we claim the following:  $v_n - T_{i-1} v_n \rightarrow 0, u_n - v_n \rightarrow 0, w_n - v_n \rightarrow 0, u_{n+1} - v_n \rightarrow 0,$

$u_{n+1} - v_n \rightarrow 0, u_{n+1} - u_n \rightarrow 0, u_{n+1} - T_{i-1} u_n \rightarrow 0$  as  $n \rightarrow \infty$ .

We have

$$\begin{aligned} \|v_n - x\|^2 &= \|u_n + \varepsilon_n(u_n - u_{n-1}) - x\|^2 \\ \|v_n - x\|^2 &\leq \|u_n - x\|^2 + 2\varepsilon_n \|u_n - x\| \|u_n - u_{n-1}\| + \varepsilon_n^2 \|u_n - u_{n-1}\|^2. \end{aligned} \tag{3.4}$$

Also, we have

$$\begin{aligned} \|w_n - x\|^2 &= \left\| \sum_{i=1}^{K+1} \gamma_{n,i} T_{i-1} v_n - x \right\|^2 \\ &= \left\| \sum_{i=1}^{K+1} \gamma_{n,i} (T_{i-1} v_n - x) \right\|^2. \end{aligned}$$

By lemma 2.4 we have

$$\begin{aligned} \|w_n - x\|^2 &\leq \sum_{i=1}^{K+1} \gamma_{n,i} \|T_{i-1} v_n - x\|^2 - \gamma_{n,0} \gamma_{n,i} g(\|v_n - T_{i-1} v_n\|) \\ &\leq \sum_{i=1}^{K+1} \gamma_{n,i} \|v_n - x\|^2 - \gamma_{n,0} \gamma_{n,i} g(\|v_n - T_{i-1} v_n\|) \\ \|w_n - x\|^2 &\leq \|v_n - x\|^2 - \gamma_{n,0} \gamma_{n,i} g(\|v_n - T_{i-1} v_n\|). \end{aligned} \tag{3.5}$$

Now from the scheme (3.0), we have

$$\begin{aligned} \|u_{n+1} - x\|^2 &= \|\alpha_n f(u_n) + \beta_n u_n + \omega_n w_n - x\|^2 \\ &= \|\alpha_n (f(u_n) - x) + \beta_n (u_n - x) + \omega_n (w_n - x)\|^2 \\ &\leq \alpha_n \|f(u_n) - x\|^2 + \beta_n \|u_n - x\|^2 + \omega_n \|w_n - x\|^2. \end{aligned}$$

Using equations (3.4) and (3.5) we obtain

$$\begin{aligned} \|u_{n+1} - x\|^2 &\leq \alpha_n \|f(u_n) - x\|^2 + \beta_n \|u_n - x\|^2 + \omega_n [\|v_n - x\|^2 - \gamma_{n,0} \gamma_{n,i} g(\|v_n - T_{i-1} v_n\|)] \\ &\leq \alpha_n \|f(u_n) - x\|^2 + \beta_n \|u_n - x\|^2 + \omega_n \|v_n - x\|^2 - \omega_n \gamma_{n,0} \gamma_{n,i} g(\|v_n - T_{i-1} v_n\|) \\ &\leq \alpha_n \|f(u_n) - x\|^2 + \beta_n \|u_n - x\|^2 - \omega_n \gamma_{n,0} \gamma_{n,i} g(\|v_n - T_{i-1} v_n\|) \\ &\quad + \omega_n [\|u_n - x\|^2 + 2\varepsilon_n \|u_n - x\| \|u_n - u_{n-1}\| + \varepsilon_n^2 \|u_n - u_{n-1}\|^2] \\ &= \alpha_n \|f(u_n) - x\|^2 + \beta_n \|u_n - x\|^2 + \omega_n \|u_n - x\|^2 - \omega_n \gamma_{n,0} \gamma_{n,i} g(\|v_n - T_{i-1} v_n\|) \\ &\quad + \omega_n [2\varepsilon_n \|u_n - x\| \|u_n - u_{n-1}\| + \varepsilon_n^2 \|u_n - u_{n-1}\|^2] \\ &= \alpha_n \|f(u_n) - x\|^2 + (\beta_n + \omega_n) \|u_n - x\|^2 - \omega_n \gamma_{n,0} \gamma_{n,i} g(\|v_n - T_{i-1} v_n\|) \\ &\quad + \omega_n \varepsilon_n \|u_n - u_{n-1}\| [2\|u_n - x\| + \varepsilon_n \|u_n - u_{n-1}\|]. \\ \|u_{n+1} - x\|^2 &\leq \alpha_n \|f(u_n) - x\|^2 + (1 - \alpha_n) \|u_n - x\|^2 - \omega_n \gamma_{n,0} \gamma_{n,i} g(\|v_n - T_{i-1} v_n\|) \\ &\quad + \omega_n \varepsilon_n \|u_n - u_{n-1}\| [2\|u_n - x\| + \varepsilon_n \|u_n - u_{n-1}\|]. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \omega_n \gamma_{n,0} \gamma_{n,i} g(\|v_n - T_{i-1} v_n\|) &\leq \alpha_n \|f(u_n) - x\|^2 + \|u_n - x\|^2 - \alpha_n \|u_n - x\|^2 - \|u_{n+1} - x\|^2 \\ &\quad + \omega_n \varepsilon_n \|u_n - u_{n-1}\| [2\|u_n - x\| + \varepsilon_n \|u_n - u_{n-1}\|] \\ \omega_n \gamma_{n,0} \gamma_{n,i} g(\|v_n - T_{i-1} v_n\|) &\leq \alpha_n (\|f(u_n) - x\|^2 - \|u_n - x\|^2) - (\|u_{n+1} - x\|^2 - \|u_n - x\|^2) \\ &\quad + \omega_n \varepsilon_n \|u_n - u_{n-1}\| [2\|u_n - x\| + \varepsilon_n \|u_n - u_{n-1}\|] \end{aligned}$$

By taking limits of both side, assumptions 2, 3, 5, the boundedness of  $\gamma_{n,0} \gamma_{n,i}$  and properties of “g” we have

$$\lim_{n \rightarrow \infty} \|v_n - T_{i-1} v_n\| = 0. \tag{3.6}$$

Also, we obtain

$$\begin{aligned} \|v_n - u_n\| &= \|u_n + \varepsilon_n (u_n - u_{n-1}) - u_n\| \\ &\leq \varepsilon_n \|u_n - u_{n-1}\| \\ \lim_{n \rightarrow \infty} \|v_n - u_n\| &\leq \lim_{n \rightarrow \infty} \varepsilon_n \|u_n - u_{n-1}\| = 0. \end{aligned} \tag{3.7}$$

We also have

$$\begin{aligned} \|w_n - x\|^2 &= \left\| \sum_{i=1}^{K+1} \gamma_{n,i} T_{i-1} v_n - x \right\|^2 \\ &= \left( \left\| \sum_{i=1}^{K+1} \gamma_{n,i} T_{i-1} v_n - x \right\| \right)^2 \\ &\leq \sum_{i=1}^{K+1} \gamma_{n,i} \|T_{i-1} v_n - x\|^2 \\ &\leq \sum_{i=1}^{K+1} \gamma_{n,i} \|v_n - x\|^2 \\ &= \|v_n - x\|^2. \end{aligned}$$

Using equation(3.4), we have

$$\|w_n - x\|^2 \leq \|u_n - x\|^2 + 2\varepsilon_n \|u_n - x\| \|u_n - u_{n-1}\| + \varepsilon_n^2 \|u_n - u_{n-1}\|^2 \tag{3.8}$$

Using lemma 2.3, and equation(3.8), we have

$$\begin{aligned} \|u_{n+1} - x\|^2 &= \|\alpha_n f(u_n) + \beta_n u_n + \omega_n w_n - x\|^2 \\ &= \|\alpha_n (f(u_n) - x) + \beta_n (u_n - x) + \omega_n (w_n - x)\|^2 \\ &\leq \alpha_n \|f(u_n) - x\|^2 + \beta_n \|u_n - x\|^2 + \omega_n \|w_n - x\|^2 - \beta_n \omega_n g(\|u_n - w_n\|) \\ &\leq \alpha_n \|f(u_n) - x\|^2 + \beta_n \|u_n - x\|^2 + \omega_n [\|u_n - x\|^2 + 2\varepsilon_n \|u_n - x\| \|u_n - u_{n-1}\| + \varepsilon_n^2 \|u_n - u_{n-1}\|^2] \\ &\quad - \beta_n \omega_n g(\|u_n - w_n\|) \\ &= \alpha_n \|f(u_n) - x\|^2 + \beta_n \|u_n - x\|^2 + \omega_n \|u_n - x\|^2 + \omega_n [2\varepsilon_n \|u_n - x\| \|u_n - u_{n-1}\| + \varepsilon_n^2 \|u_n - u_{n-1}\|^2] \\ &\quad - \beta_n \omega_n g(\|u_n - w_n\|) \\ &= \alpha_n \|f(u_n) - x\|^2 + \beta_n \|u_n - x\|^2 + \omega_n \|u_n - x\|^2 + \omega_n \varepsilon_n \|u_n - u_{n-1}\| [2\|u_n - x\| + \varepsilon_n \|u_n - u_{n-1}\|] \\ &\quad - \beta_n \omega_n g(\|u_n - w_n\|) \end{aligned}$$

$$\|u_{n+1} - x\|^2 \leq \alpha_n \|f(u_n) - x\|^2 + (1 - \alpha_n) \|u_n - x\|^2 + \omega_n \varepsilon_n \|u_n - u_{n-1}\| [2\|u_n - x\| + \varepsilon_n \|u_n - u_{n-1}\|] - \beta_n \omega_n g(\|u_n - w_n\|).$$

Furthermore, we have

$$\begin{aligned} \beta_n \omega_n g(\|u_n - w_n\|) &\leq \alpha_n \|f(u_n) - x\|^2 + \|u_n - x\|^2 - \alpha_n \|u_{n+1} - x\|^2 \\ &\quad + \omega_n \varepsilon_n \|u_n - u_{n-1}\| [2\|u_n - x\| + \varepsilon_n \|u_n - u_{n-1}\|] \\ \beta_n \omega_n g(\|u_n - w_n\|) &\leq \alpha_n (\|f(u_n) - x\|^2 - \|u_n - x\|^2) - (\|u_{n+1} - x\|^2 - \|u_n - x\|^2) \\ &\quad + \omega_n \varepsilon_n \|u_n - u_{n-1}\| [2\|u_n - x\| + \varepsilon_n \|u_n - u_{n-1}\|]. \end{aligned}$$

By taking limits of both side, assumptions 2,3,4,5 and the properties of “g” we have

$$\lim_{n \rightarrow \infty} \|u_n - w_n\| = 0. \tag{3.9}$$

We also have

$$\begin{aligned} \|w_n - v_n\| &= \|w_n - u_n + u_n - v_n\| \\ &\leq \|w_n - u_n\| + \|u_n - v_n\|. \end{aligned}$$

By(3.7) and (3.9), we have

$$\lim_{n \rightarrow \infty} \|w_n - v_n\| = 0. \tag{3.10}$$

Consequently, we have

$$\begin{aligned} \|u_{n+1} - v_n\| &= \|\alpha_n f(u_n) + \beta_n u_n + \omega_n w_n - v_n\| \\ &= \|\alpha_n (f(u_n) - v_n) + \beta_n (u_n - v_n) + \omega_n (w_n - v_n)\| \\ &\leq \alpha_n \|f(u_n) - v_n\| + \beta_n \|u_n - v_n\| + \omega_n \|w_n - v_n\|. \end{aligned}$$

By condition 1 and (3.7) with (3.10), we have

$$\lim_{n \rightarrow \infty} \|u_{n+1} - v_n\| = 0. \tag{3.11}$$

We also obtain the following

$$\begin{aligned} \|u_{n+1} - u_n\| &= \|u_{n+1} - v_n + v_n - u_n\| \\ &\leq \|u_{n+1} - v_n\| + \|v_n - u_n\|. \end{aligned}$$

Using (3.11) and (3.7), we have

$$\lim_{n \rightarrow \infty} \|u_{n+1} - u_n\| = 0. \tag{3.12}$$

At the end, we have

$$\begin{aligned} \|u_n - T_{i-1}u_n\| &= \|u_n - u_{n+1} + u_{n+1} - v_n + v_n - T_{i-1}v_n + T_{i-1}v_n - T_{i-1}u_n\| \\ &\leq \|u_n - u_{n+1}\| + \|u_{n+1} - v_n\| + \|v_n - T_{i-1}v_n\| + \|T_{i-1}v_n - T_{i-1}u_n\| \\ &\leq \|u_n - u_{n+1}\| + \|u_{n+1} - v_n\| + \|v_n - T_{i-1}v_n\| + \|v_n - u_n\| \end{aligned}$$

By equations(3.6), (3.7), (3.11) and (3.12), we have

$$\lim_{n \rightarrow \infty} \|u_n - T_{i-1}u_n\| = 0.$$

Since  $\lim_{n \rightarrow \infty} \|u_n - T_{i-1}u_n\| = 0$  we conclude that

$$\lim_{n \rightarrow \infty} \|u_n - T_i u_n\| = 0. \quad (3.13)$$

The set of all weak limit points of the bounded family  $(u_n)$  is denoted by  $\theta_m(u_n)$ . Since

$X$  is reflexive and  $(u_n)$  is bounded, there is a subsequence  $u_{n_k}$  from  $(u_n)$  that goes weakly towards a point  $u$  consequently since  $\lim_{n \rightarrow \infty} \|u_n - T_i u_n\| = 0$  and by Lemma 2.1, it follows that  $\lim_{n \rightarrow \infty} \|u_{n_k} - T_i u_{n_k}\| = 0$ , then the map is demiclosed at zero, we have  $Tx = x$ , therefore  $\theta_m(u_n) \subseteq A(T_i)$ . We demonstrate that  $\theta_m(u_n)$  is a single element set in order to demonstrate that  $(u_n)$  converges weakly to  $x \in A(T_i)$ . Let  $X$  satisfies opial's property, suppose  $x, r \in \theta_m(u_n)$  and let  $u_{n_s}$  and  $u_{n_t}$  are subsequences belonging to  $u_n$  for which  $u_{n_s} \rightharpoonup x$  as well as  $u_{n_t} \rightharpoonup r$ . Then for  $x \neq r$ , we provide

$$\liminf_{s \rightarrow \infty} \|u_{n_s} - r\| > \liminf_{s \rightarrow \infty} \|u_{n_s} - x\|. \quad (3.14)$$

$$\liminf_{t \rightarrow \infty} \|u_{n_t} - x\| > \liminf_{t \rightarrow \infty} \|u_{n_t} - r\|. \quad (3.15)$$

Now by (3.14) and (3.15) we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|u_n - x\| &= \liminf_{s \rightarrow \infty} \|u_{n_s} - x\| \\ &< \liminf_{s \rightarrow \infty} \|u_{n_s} - r\| \\ &= \lim_{n \rightarrow \infty} \|u_n - r\| \\ &= \liminf_{t \rightarrow \infty} \|u_{n_t} - r\| \\ &< \liminf_{t \rightarrow \infty} \|u_{n_t} - x\| \\ &= \lim_{n \rightarrow \infty} \|u_n - x\| \end{aligned}$$

The contradiction implies that the  $\theta_m(u_n)$  contains only one element, hence  $u_n \rightharpoonup x$ .

#### 4. Conclusion

In this paper, we proposed a viscosity–inertial iterative scheme for approximating attractive points of finitely many generalized nonexpansive mappings in a uniformly convex Banach space. Under suitable control conditions, we proved that the generated sequence converges weakly to a common attractive point of all mappings. This result extends the work of Thongphaen et al. (2022) that was restricted to two mappings and provides a more general framework for analyzing convergence in nonlinear settings. The findings contribute to the development of iterative methods in fixed point theory and may inspire further research toward achieving strong convergence and exploring wider classes of mappings.

#### Conflict of Interest

The author declares that there is no conflict of interest.

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