



## Article Info

Received: 6<sup>th</sup> February 2020

Revised: 8<sup>th</sup> May 2020

Accepted: 10<sup>th</sup> May 2020

<sup>1</sup>Basic Sciences Department, Federal College of Freshwater Fisheries Technology, New Bussa, Niger, Nigeria

<sup>2</sup>Department of Mathematics, Modibbo Adama University of Technology, Yola, Nigeria

<sup>3</sup>Department of Mathematical Sciences, Federal University Lokoja, Kogi State, Nigeria

\*Corresponding author's email:

[malqwisa@gmail.com](mailto:malqwisa@gmail.com)

Cite this: *CaJoST*, 2020, 2, 81-88

## Four and Five-Point Integrators for the Solution of Systems of Stiff Initial Value Problems

Samuel Adamu<sup>1\*</sup>, Musa Kida<sup>2</sup>, Dahiru Umar<sup>2</sup> and Kefas Bitrus<sup>3</sup>

Four and five-point second derivative hybrid block integrators were developed for the solution of stiff Ordinary Differential Equations (ODEs) using polynomial approximate solution. Collocation and interpolation technique was adopted in the development of the methods where collocation points were determined using standard collocation method. The stability properties of the new integrators were established and the efficiency of the integrators were tested on some numerical examples with a written program using MATLAB 8.5, where *g* – values were computed in addition to computation of *f* – values at the internal stages. The results show that, the new integrators are efficient for the solution of stiff Initial Value Problems (IVPs) of first order ODEs.

**Keywords:** Second derivative, 4-point block integrator, 5-point block integrator, Stiff initial value problems

## 1. Introduction

This paper considers the solution of Initial Value Problems (IVPs) for first order Ordinary Differential Equations (ODEs) of the form:

$$y' = f(x, y(x)), \quad y(a) = y_0, \quad x \in [a, b] \quad (1.1)$$

where  $f : [a, b] \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ ,  $y_0 \in \mathbb{R}^m$  is continuous and differentiable and  $f$  is assumed to satisfy the existence and uniqueness theorem.

ODEs are applied in a number of important areas, including chemical reactions, control systems, and electronic networks which lead to systems of differential equations, see [1], [2], [3]. These equations are sometimes difficult to solve because different processes in the system behave with significantly different time scales. Problems with solution components containing widely different time scales are said to be stiff problems [1], [4].

Hybrid technique and Second derivative terms were introduced to derive integrators which are highly stable and efficient for the solution of differential systems, see [3], [5] for details. Hybrid methods involve the evaluation of the differential system at off-step points. Second derivative methods involve the computation of *g* – values in addition to computation of *f* – values at the internal stages, where *g* is defined by

$$y'' = g(y(x)), \quad g_i(y(x)) = \sum_{j=1}^k \frac{\partial f_i(y(x))}{\partial y_j} f_j(y(x)), \quad (1.2)$$

where  $i = 1, 2, \dots, k$  [5].

Several authors studied more than one-point block methods which include but not limited to [1], [6], [7], [8], and reported that, higher point block methods are efficient for the solution of initial value problems of ordinary differential equations.

This paper therefore, is concerned with the development of hybrid second derivative four and five points collocation integrators for the solution of IVPs ODEs. These integrators will combine the efficiency of single step methods and the high order of accuracy of multistep methods.

## 2. Materials and Methods

### Mathematical Background [3]

(i) Consider the approximate solution

$$y(x) = \sum_{n=0}^k \alpha_n x^n \quad (2.1)$$

where  $x \in [a, b]$ ,  $\alpha_n \in \mathbb{R}$  are the unknown parameters to be determined. To generate the collocation points, we considered the standard collocation method

$$x_k = \xi + \frac{(\beta - \xi)}{N}k, \quad k = 1, 2, \dots, N, \quad [\xi, \beta] = [0, 1] \quad (2.2)$$

(ii) Evaluate the first and second derivative of equation (2.1) at  $x = \{x_{n+j} \mid j = 0, \dots, k\}$ , to give the system

$$XA = U \quad (2.3)$$

where

$$A = [a_0, \dots, a_k], \quad U = [y_n, \dots, y_{n+r}, f_n, \dots, f_{n+s}, g_n, \dots, g_{n+\tau}]^T.$$

(iii) Impose the following conditions

$$\begin{aligned} y(x_{n+j}) &= y_{n+j}, & j &= 0, 1, \dots, r \\ y'(x_{n+j}) &= f_{n+j}, & j &= 0, 1, \dots, s \\ y''(x_{n+j}) &= g_{n+j}, & j &= 0, 1, \dots, \tau \end{aligned}$$

on equation (2.1) to give

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & \Lambda & x_n^k \\ M & M & M & O & M \\ 1 & x_{n+r} & x_{n+r}^2 & \Lambda & x_{n+r}^k \\ 0 & 1 & 2x_n & \Lambda & kx_n^{k-1} \\ M & M & M & O & M \\ 0 & 1 & 2x_{n+s} & \dots & kx_{n+s}^{k-1} \\ 0 & 0 & 2 & \dots & k(k-1)x_n^{k-2} \\ M & M & M & O & M \\ 0 & 0 & 2 & \dots & k(k-1)x_{n+\tau}^{k-2} \end{bmatrix}$$

where  $k = r + s + \tau - 1$ .

(iv) Solve (2.2) for the unknown parameters using Cramer's rule and substitute the results into (2.1), to give

$$y_{n+t} = \sum_{j=0}^r \alpha_j(t)y_{n+j} + h \sum_{j=0}^s \beta_j(t)f_{n+j} + h^2 \sum_{j=0}^{\tau} \sigma_j(t)g_{n+j} \quad (2.4)$$

where  $\alpha_j(t)$ ,  $\beta_j(t)$ , and  $\sigma_j(t)$  are polynomial

of degree  $r + s + \tau - 1$  and  $t = \frac{x - x_n}{h}$ .

**Fifth order 4-point integrator**

(i) Evaluate (2.1) at point  $x_n$ , first derivative of (2.1) at  $0(\frac{1}{4})\frac{3}{4}$  for the first point,  $0, \frac{1}{4}, \frac{1}{2}, 1$  for the second point,  $0, \frac{1}{4}, \frac{3}{4}, 1$  for the third point,  $0, \frac{1}{2}, \frac{3}{4}, 1$  for the fourth point and second derivative of (2.1) at  $x_{n+1}$  for the continuous scheme. Evaluating the continuous scheme at  $t = \frac{1}{4}(\frac{1}{4})1$  to give the discrete scheme

$$y_{n+\frac{1}{2}} = y_n + \frac{1583}{18000}hf_n + \frac{2641}{12000}hf_{n+\frac{1}{4}} - \frac{493}{6000}hf_{n+\frac{1}{2}} + \frac{869}{36000}hf_{n+\frac{3}{4}} - \frac{19}{24000}h^2g_{n+1} \quad (2.5)$$

$$y_{n+\frac{1}{2}} = y_n + \frac{13}{160}hf_n + \frac{46}{135}hf_{n+\frac{1}{4}} + \frac{3}{40}hf_{n+\frac{1}{2}} + \frac{13}{4320}hf_{n+\frac{3}{4}} - \frac{1}{1440}h^2g_{n+1} \quad (2.6)$$

$$y_{n+\frac{3}{4}} = y_n + \frac{21}{320}hf_n + \frac{67}{160}hf_{n+\frac{1}{4}} + \frac{69}{160}hf_{n+\frac{1}{2}} - \frac{53}{320}hf_{n+\frac{3}{4}} + \frac{3}{160}h^2g_{n+1} \quad (2.7)$$

$$y_{n+1} = y_n + \frac{13}{90}hf_n + \frac{14}{15}hf_{n+\frac{1}{2}} - \frac{32}{45}hf_{n+\frac{3}{4}} + \frac{19}{30}hf_{n+1} - \frac{1}{15}h^2g_{n+1} \quad (2.8)$$

(ii) writing (2.5) - (2.8) in block, to get

$$A^{(1)}Y_{m+1} = A^{(0)}Y_m + hB^{(0)}F_m + hB^{(1)}F_{m+1} + h^2\gamma^{(1)}G_{m+1} \quad (2.9)$$

where

$$Y_{m+1} = [y_{n+\frac{1}{4}}, y_{n+\frac{1}{2}}, y_{n+\frac{3}{4}}, y_{n+1}]^T, Y_m = [y_{n-1}, y_{n-2}, y_{n-3}, y_n]^T,$$

$$F_{m+1} = [f_{n+\frac{1}{4}}, f_{n+\frac{1}{2}}, f_{n+\frac{3}{4}}, f_{n+1}]^T, F_m = [f_{n-1}, f_{n-2}, f_{n-3}, f_n]^T,$$

$$G_{m+1} = [g_{n+\frac{1}{4}}, g_{n+\frac{1}{2}}, g_{n+\frac{3}{4}}, g_{n+1}]^T,$$

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B^{(0)} = \begin{bmatrix} 0 & 0 & 0 & \frac{1583}{18000} \\ 0 & 0 & 0 & \frac{13}{160} \\ 0 & 0 & 0 & \frac{21}{320} \\ 0 & 0 & 0 & \frac{13}{90} \end{bmatrix},$$

$$B^{(1)} = \begin{bmatrix} \frac{2641}{12000} & -\frac{493}{6000} & \frac{869}{36000} & 0 \\ \frac{46}{135} & \frac{3}{40} & 0 & \frac{13}{4320} \\ \frac{67}{160} & 0 & \frac{69}{160} & -\frac{53}{320} \\ 0 & \frac{14}{15} & -\frac{32}{45} & \frac{19}{30} \end{bmatrix}, \gamma^{(1)} = \begin{bmatrix} 0 & 0 & 0 & -\frac{19}{24000} \\ 0 & 0 & 0 & -\frac{1}{1440} \\ 0 & 0 & 0 & \frac{3}{160} \\ 0 & 0 & 0 & -\frac{1}{15} \end{bmatrix}$$

(iii) The coefficients of the second derivative collocation integrator (2.9) can also be represented more compactly in Butcher tableau as follows:

	0	0	0	0	0	0	0	0	0
$\frac{1}{4}$	$\frac{1583}{18000}$	$\frac{2641}{12000}$	$-\frac{493}{6000}$	$\frac{869}{36000}$	0	0	0	0	$-\frac{19}{24000}$
$\frac{2}{4}$	$\frac{13}{160}$	$\frac{46}{135}$	$\frac{3}{40}$	0	$\frac{13}{4320}$	0	0	0	$-\frac{1}{1440}$
$\frac{3}{4}$	$\frac{21}{320}$	$\frac{67}{160}$	0	$\frac{69}{160}$	$-\frac{53}{320}$	0	0	0	$\frac{3}{160}$
1	$\frac{13}{90}$	0	$\frac{14}{15}$	$-\frac{32}{45}$	$\frac{19}{30}$	0	0	0	$-\frac{1}{15}$
	$\frac{13}{90}$	0	$\frac{14}{15}$	$-\frac{32}{45}$	$\frac{19}{30}$	0	0	0	$-\frac{1}{15}$

**Sixth order 4-point integrator**

(i) Evaluate (2.1) at point  $x_n$ , first derivative of (2.1) at  $0(\frac{1}{4})1$  and second derivative of (2.1) at  $x_{n+1}$  for the first point,  $x_{n+\frac{3}{4}}$  for the second point,

$x_{n+\frac{2}{4}}$  for the third point, and  $x_{n+\frac{1}{4}}$  for the fourth point, to get the continuous scheme. Evaluating the continuous scheme at  $t = \frac{1}{4}(\frac{1}{4})1$  to give the discrete scheme

$$y_{n+\frac{2}{4}} = y_n + \frac{1873}{23040}hf_n + \frac{23}{90}hf_{n+\frac{1}{4}} - \frac{311}{1920}hf_{n+\frac{2}{4}} + \frac{47}{360}hf_{n+\frac{3}{4}} - \frac{1277}{23040}hf_{n+1} + \frac{3}{512}h^2g_{n+1} \tag{2.10}$$

$$y_{n+\frac{3}{4}} = y_n + \frac{41}{540}hf_n + \frac{67}{180}hf_{n+\frac{1}{4}} - \frac{1}{60}hf_{n+\frac{2}{4}} + \frac{31}{540}hf_{n+\frac{3}{4}} + \frac{1}{90}hf_{n+1} - \frac{1}{72}h^2g_{n+1} \tag{2.11}$$

$$y_{n+\frac{4}{4}} = y_n + \frac{93}{1280}hf_n + \frac{33}{80}hf_{n+\frac{1}{4}} + \frac{9}{40}hf_{n+\frac{2}{4}} + \frac{3}{80}hf_{n+\frac{3}{4}} + \frac{3}{1280}hf_{n+1} + \frac{9}{256}h^2g_{n+1} \tag{2.12}$$

$$y_{n+1} = y_n + \frac{7}{90}hf_n + \frac{16}{45}hf_{n+\frac{1}{4}} + \frac{2}{15}hf_{n+\frac{2}{4}} + \frac{16}{45}hf_{n+\frac{3}{4}} + \frac{7}{90}hf_{n+1} + 0h^2g_{n+1} \tag{2.13}$$

(ii) writing (2.10) - (2.13) in block to get

$$A^{(1)}Y_{m+1} = A^{(0)}Y_m + hB^{(0)}F_m + hB^{(1)}F_{m+1} + h^2\gamma^{(1)}G_{m+1} \tag{2.14}$$

where

$$Y_{m+1} = [y_{n+\frac{1}{4}} \ y_{n+\frac{2}{4}} \ y_{n+\frac{3}{4}} \ y_{n+1}]^T, Y_m = [y_{n-1} \ y_{n-2} \ y_{n-3} \ y_n]^T,$$

$$F_{m+1} = [f_{n+\frac{1}{4}} \ f_{n+\frac{2}{4}} \ f_{n+\frac{3}{4}} \ f_{n+1}]^T, F_m = [f_{n-1} \ f_{n-2} \ f_{n-3} \ f_n]^T,$$

$$G_{m+1} = [g_{n+\frac{1}{4}} \ g_{n+\frac{2}{4}} \ g_{n+\frac{3}{4}} \ g_{n+1}]^T,$$

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B^{(0)} = \begin{bmatrix} 0 & 0 & 0 & \frac{1873}{23040} \\ 0 & 0 & 0 & \frac{41}{540} \\ 0 & 0 & 0 & \frac{93}{1280} \\ 0 & 0 & 0 & \frac{7}{90} \end{bmatrix},$$

$$B^{(1)} = \begin{bmatrix} \frac{23}{90} & -\frac{311}{1920} & \frac{47}{360} & -\frac{1277}{23040} \\ \frac{67}{180} & -\frac{1}{60} & \frac{31}{540} & \frac{1}{90} \\ \frac{33}{80} & \frac{9}{40} & \frac{3}{80} & \frac{3}{1280} \\ \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} \end{bmatrix}, \gamma^{(1)} = \begin{bmatrix} 0 & 0 & 0 & \frac{3}{512} \\ 0 & 0 & -\frac{1}{72} & 0 \\ 0 & \frac{9}{256} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(iii) The coefficients of the second derivative collocation integrator (2.14) can also be represented more compactly in Butcher tableau as follows:

	0	0	0	0	0	0	0	0	0	0	0	0	0
$\frac{1}{4}$	$\frac{1873}{23040}$	$\frac{23}{90}$	$-\frac{311}{1920}$	$\frac{47}{360}$	$-\frac{1277}{23040}$	$\frac{3}{512}$	0	0	0	0	0	0	0
$\frac{2}{4}$	$\frac{41}{540}$	$\frac{67}{180}$	$-\frac{1}{60}$	$\frac{31}{540}$	$\frac{1}{90}$	$-\frac{1}{72}$	0	0	0	$-\frac{1}{72}$	0	0	0
$\frac{3}{4}$	$\frac{93}{1280}$	$\frac{33}{80}$	$\frac{9}{40}$	$\frac{3}{80}$	$\frac{3}{1280}$	0	0	0	$\frac{9}{256}$	0	0	0	0
1	$\frac{7}{90}$	$\frac{16}{45}$	$\frac{2}{15}$	$\frac{16}{45}$	$\frac{7}{90}$	0	0	0	0	0	0	0	0
$\frac{7}{90}$	$\frac{16}{45}$	$\frac{2}{15}$	$\frac{16}{45}$	$\frac{7}{90}$	0	0	0	0	0	0	0	0	0

**Sixth order 5-point integrator**

(i) Evaluate (2.1) at point  $x_n$ , first derivative of (2.1) at  $0(\frac{1}{5})\frac{4}{5}$  for the first point,  $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, 1$  for the second point,  $0, \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, 1$  for the third point,  $0, \frac{1}{5}, \frac{3}{5}, \frac{4}{5}, 1$  for the fourth point,  $0, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$  for the fifth point and second derivative of (2.1) at  $x_{n+1}$  for the continuous scheme. Evaluating the continuous scheme at  $t = \frac{1}{5}(\frac{1}{5})1$  to give the discrete schemes

$$y_{n+\frac{1}{5}} = y_n + \frac{65399}{986400}hf_n + \frac{96737}{493200}hf_{n+\frac{1}{5}} - \frac{17321}{164400}hf_{n+\frac{2}{5}} + \frac{28967}{493200}hf_{n+\frac{3}{5}} - \frac{15601}{986400}hf_{n+\frac{4}{5}} + \frac{9}{27400}h^2g_{n+1} \tag{2.15}$$

$$y_{n+\frac{2}{5}} = y_n + \frac{353}{5625}hf_n + \frac{17}{60}hf_{n+\frac{1}{5}} + \frac{1}{25}hf_{n+\frac{2}{5}} + \frac{4}{225}hf_{n+\frac{3}{5}} - \frac{29}{7500}hf_{n+1} + \frac{1}{1875}h^2g_{n+1} \tag{2.16}$$

$$y_{n+\frac{3}{5}} = y_n + \frac{1389}{20000}hf_n + \frac{381}{1600}hf_{n+\frac{1}{5}} + \frac{19}{80}hf_{n+\frac{2}{5}} + \frac{93}{800}hf_{n+\frac{3}{5}} - \frac{2453}{40000}hf_{n+1} + \frac{57}{10000}h^2g_{n+1} \tag{2.17}$$

$$y_{n+\frac{4}{5}} = y_n + \frac{314}{5625}hf_n + \frac{73}{225}hf_{n+\frac{1}{5}} + \frac{4}{9}hf_{n+\frac{2}{5}} - \frac{22}{225}hf_{n+\frac{3}{5}} + \frac{137}{1875}hf_{n+1} - \frac{4}{625}h^2g_{n+1} \tag{2.18}$$

$$y_{n+1} = y_n + \frac{31}{288}hf_n + \frac{125}{144}hf_{n+\frac{2}{5}} - \frac{125}{144}hf_{n+\frac{3}{5}} + \frac{125}{96}hf_{n+\frac{4}{5}} - \frac{59}{144}hf_{n+1} + \frac{1}{24}h^2g_{n+1} \tag{2.19}$$

(ii) writing (2.15) - (2.19) in block to get

$$A^{(1)}Y_{m+1} = A^{(0)}Y_m + hB^{(0)}F_m + hB^{(1)}F_{m+1} + h^2\gamma^{(1)}G_{m+1} \tag{2.20}$$

where

$$Y_{m+1} = [y_{n+\frac{1}{5}} \ y_{n+\frac{2}{5}} \ y_{n+\frac{3}{5}} \ y_{n+\frac{4}{5}} \ y_{n+1}]^T, Y_m = [y_{n-1} \ y_{n-2} \ y_{n-3} \ y_{n-4} \ y_n]^T,$$

$$F_{m+1} = [f_{n+\frac{1}{5}} \ f_{n+\frac{2}{5}} \ f_{n+\frac{3}{5}} \ f_{n+\frac{4}{5}} \ f_{n+1}]^T, F_m = [f_{n-1} \ f_{n-2} \ f_{n-3} \ f_{n-4} \ f_n]^T,$$

$$G_{m+1} = [g_{n+\frac{1}{5}} \ g_{n+\frac{2}{5}} \ g_{n+\frac{3}{5}} \ g_{n+\frac{4}{5}} \ g_{n+1}]^T,$$

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B^{(1)} = \begin{bmatrix} \frac{96737}{493200} & -\frac{17321}{164400} & \frac{28967}{493200} & -\frac{15601}{986400} & 0 \\ \frac{17}{60} & \frac{1}{25} & \frac{4}{225} & 0 & -\frac{29}{7500} \\ \frac{381}{1600} & \frac{19}{80} & 0 & \frac{93}{800} & -\frac{2453}{40000} \\ \frac{73}{225} & 0 & \frac{4}{9} & -\frac{22}{225} & \frac{137}{1875} \\ 0 & \frac{125}{144} & -\frac{125}{144} & \frac{125}{96} & -\frac{59}{144} \end{bmatrix},$$



$$B^{(1)} = \begin{bmatrix} \frac{4919}{22400} & -\frac{6347}{37800} & \frac{2563}{16800} & -\frac{307}{2800} & \frac{129571}{3024000} \\ \frac{1447}{4725} & -\frac{29}{1050} & \frac{26}{175} & -\frac{2909}{37800} & -\frac{1}{105} \\ \frac{3501}{11200} & -\frac{9}{700} & \frac{34}{175} & \frac{9}{175} & -\frac{9}{2240} \\ \frac{176}{525} & \frac{664}{4725} & \frac{32}{175} & \frac{46}{525} & -\frac{16}{4725} \\ \frac{2725}{24192} & \frac{25}{56} & \frac{25}{672} & \frac{925}{3024} & \frac{53}{896} \end{bmatrix},$$

$$\gamma^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{863}{252000} \\ 0 & 0 & 0 & \frac{37}{3150} & 0 \\ 0 & 0 & -\frac{87}{2800} & 0 & 0 \\ 0 & \frac{32}{1575} & 0 & 0 & 0 \\ -\frac{55}{2016} & 0 & 0 & 0 & 0 \end{bmatrix}$$

(iii) The coefficients of the second derivative collocation integrator (2.26) can also be represented more compactly in Butcher tableau as follows:

$$\begin{array}{c|c|c} c & A & \tilde{A} \\ \hline & b^T & \tilde{b} \end{array} \quad \square$$

0	0	0	0	0	0	0	0	0	0	0	<del>863</del> 252000	
$\frac{1}{5}$	<del>2627</del> 42000	<del>4919</del> 22400	<del>6347</del> 37800	<del>2563</del> 16800	<del>307</del> 2800	<del>129571</del> 3024000	0	0	0	0	$\frac{37}{3150}$	0
$\frac{2}{5}$	<del>83</del> 1400	<del>1447</del> 4725	<del>29</del> 1050	<del>26</del> 175	<del>2909</del> 37800	<del>1</del> 105	0	0	0	<del>87</del> 2800	0	0
$\frac{3}{5}$	<del>41</del> 700	<del>3501</del> 11200	<del>9</del> 700	<del>34</del> 175	<del>9</del> 175	<del>9</del> 2240	0	0	$\frac{32}{1575}$	0	0	0
$\frac{4}{5}$	<del>2</del> 35	<del>176</del> 525	<del>664</del> 4725	<del>32</del> 175	<del>46</del> 525	<del>16</del> 4725	0	<del>55</del> 2016	0	0	0	0
1	<del>13</del> 336	<del>2725</del> 24192	<del>25</del> 56	<del>25</del> 672	<del>925</del> 3024	<del>53</del> 896	0	0	0	0	0	0
	<del>13</del> 336	<del>2725</del> 24192	<del>25</del> 56	<del>25</del> 672	<del>925</del> 3024	<del>53</del> 896	0	0	0	0	0	0

**Stability Analysis**

Order and error constant of the integrators

Definition: The linear operator and the associate linear multistep methods are said to be of order  $p$  if  $c_0 = c_1 = c_2 = \dots = c_p = 0$  and  $c_{p+1} \neq 0$  is called the error constant which implies that the local truncation error is given as

$$t_{n+k} = c_{p+1}h^{p+1}(x_n) + (O)h^{p+1}$$

So, evaluating each row of (2.9) and (2.14) in a Taylor series about  $x_n$  gives

$$L[y(x), h] = y_{n+i} - \sum_{j=0}^r \alpha_j(t)y_{n+j} - h \sum_{j=0}^s \beta_j(t)f_{n+j} - h^2 \sum_{j=0}^t \sigma_j(t)g_{n+j} \tag{3.1}$$

$$h^{p+n} \neq 0$$

$$p + n = 6, 7, 7, 8$$

where  $n$  is the order of the differential equation. The first non-zero term is the error constant. Therefore, the order of the 4-point integrators developed are  $p = [5, 5, 5, 5]^T$  and

$p = [6, 6, 6, 6]^T$  with error constant

$$\left[ \frac{1277}{245760000}h^6, \frac{1}{307200}h^6, -\frac{33}{3276800}h^6, \frac{1}{19200}h^6 \right]^T \tag{3.2}$$

and

$$\left[ -\frac{337}{495452160}h^7, -\frac{23}{61931520}h^7, -\frac{1}{4587520}h^7, -\frac{1}{1935360}h^7 \right]^T \tag{3.3}$$

Respectively. Similarly, the order of the 5-point integrators developed are  $p = [6, 6, 6, 6, 6]^T$  and  $p = [7, 7, 7, 7, 7]^T$  with error constant

$$\left[ -\frac{129571}{647325000000}h^7, -\frac{227}{1476562500}h^7, -\frac{411}{875000000}h^7, \frac{86}{369140625}h^7, -\frac{19}{7560000}h^7 \right]^T \tag{3.4}$$

and

$$\left[ \frac{2633}{110250000000}h^8, \frac{1}{68906250}h^8, \frac{141}{12250000000}h^8, \frac{8}{861328125}h^8, -\frac{11}{882000000}h^8 \right]^T \tag{3.5}$$

respectively.

**Consistency of the integrators**

The orders of the integrators are  $p = 5, 6, 6, 7 > 1$ , therefore, the integrators are consistent.

**Stability of the Block integrators**

The block integrators (2.9) and (2.14) are zero stable since the roots  $z_s, s = 1, 2, 3 \dots n$  of the first characteristics polynomial  $\rho(z)$  is defined by

$$\rho(\lambda) = \det[A^{(1)}\lambda - A^{(0)}] = 0$$

$$\lambda^4 - \lambda^3 = \lambda^3(\lambda - 1) = 0$$

Solving for  $\lambda$  we have  $\lambda = [0, 0, 0, 1]^T$ . Hence, the block integrators (2.9) and (2.14) are zero stable. Similarly, the block integrator (2.20) and (2.26) are zero stable since

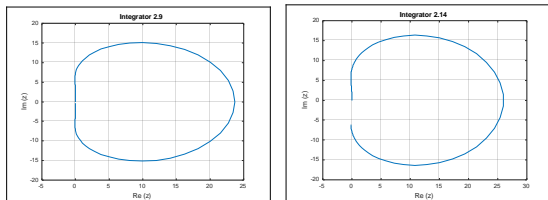
$$\lambda^4 - \lambda^3 = \lambda^3(\lambda - 1) = 0, \lambda = [0, 0, 0, 0, 1]^T.$$

**Convergence**

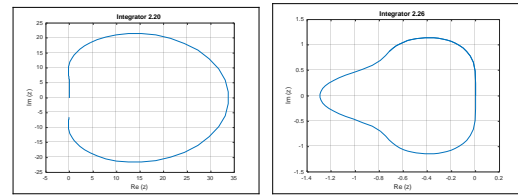
Since the new integrators (2.9), (2.14), (2.20) and (2.26) are both consistent and zero-stable, therefore, they are convergent.

**Region of Absolute Stability**

This is achieved by substituting the test equation  $y' = \lambda y$  in (2.9), (2.14), (2.20) and (2.26), and the result coded using MATLAB 8.5 to give



**Fig. 1 & 2: Regions of absolute stability of the 4-point integrators**



**Fig. 3 & 4: Regions of absolute stability of the 5-point integrator**

**3. Results and Discussion**

**Table I: Notations**

$x$	Step Size
FO4PI	Fifth order 4 - point integrator
SO4PI	Sixth order 4 - point integrator
SO5PI	Sixth order 5 - point integrator
SEO5PI	Seventh order 5 - point integrator
Er	Error

The following problems are considered to test the efficiency of the new integrators.

**Problem 1**

$$y_1' = -y_1 + 95y_2 \quad y_1(0) = 1 \quad 0 \leq x \leq 10$$

$$y_2' = -y_1 - 97y_2 \quad y_2(0) = 1$$

with exact solution

$$y(x) = \frac{1}{47} \begin{bmatrix} 95e^{-2x} - 48e^{-96x} \\ 48e^{-96x} - e^{-2x} \end{bmatrix}$$

the eigenvalues are  $-2$  and  $-96$ .

**Table II: Result of problem 1**

$h$	Err in FO4PI	Err in SO4PI	Err in SO5PI	Err in SEO5PI	Err in [1]	Err in [3]
0.02	2.2683e+02	5.8147e-01	2.88345e+10	5.6572e+03	3.06562e-002	3.78600e-002
0.03	1.1744e-02	7.4005e-02	2.9116e+20	1.5572e+02	3.93978e-002	5.65676e-003
0.04	5.1353e-03	2.5397e-02	2.9000e+30	1.2892e+02	8.39129e-003	8.59142e-003
0.05	2.2404e-03	7.4578e-03	2.9686e+40	1.0645e+02	9.02719e-004	9.53313e-004
0.06	5.2690e-03	2.7022e-03	2.9976e+50	8.7885e-03	9.09318e-002	9.98889e-004

This problem is solved by considering  $N = 100$  with  $tol = 10^{(-19)}$ . The results as presented in Table II show that, the integrator SO5PI gives better approximation than the other three integrators. Varying the collocation points for the first derivative and keeping collocation point constant for the second derivative prove to be more efficient as can be clearly seen in SO5PI and SEO5PI.

**Problem 2**

$$y_1' = -100y_1 + 9.901y_2 \quad y_1(0) = 1 \quad 0 \leq x \leq 10$$

$$y_2' = 0.1y_1 - y_2 \quad y_2(0) = 10$$

with exact solution

$$y(x) = \begin{bmatrix} e^{-0.99x} \\ 10e^{-0.99x} \end{bmatrix}$$

Eigenvalues  $-0.99$  and  $-100.01$ .

**Table III:** Result of problem 2

$h$	Err in FO4PI	Err in SO4PI	Err in SO5PI	Err in SEO5PI	Err in [1]	Err in [3]
0.02	5.1061e-01	3.1506e-01	2.07965e+05	7.6382e-01	8.35359e-002	6.12353e-003
0.03	4.0654e-01	2.1469e-01	1.92500e+10	8.0995e-01	9.10218e-003	7.43201e-004
0.04	2.73165e-01	8.1196e-02	1.8103e+25	7.5183e-01	9.18073e-004	9.08032e-004
0.05	1.7799e-01	3.43125e-02	1.7024e+35	6.8546e-01	9.18862e-005	9.58862e-006
0.06	1.1526e-01	1.41225e-02	1.60095e+45	6.2331e-01	9.18939e-006	9.81944e-007

This problem is solved by considering  $N = 100$  with  $tol = 10^{(-19)}$ . The results as presented in Table III show that, the integrator SO5PI also gives better approximation than the other three integrator.

**Problem 3**

$$y_1' = 9y_1 + 24y_2 + 5\cos x - \frac{1}{3}\sin x \quad y_1(0) = \frac{4}{3} \quad 0 \leq x \leq 10$$

$$y_2' = -24y_1 - 51y_2 - 9\cos x + \frac{1}{3}\sin x \quad y_2(0) = \frac{2}{3}$$

with exact solution

$$y(x) = \begin{bmatrix} 2e^{-3x} - e^{-39x} + \frac{1}{3}\cos x \\ -e^{-3x} + 2e^{-39x} - \frac{1}{3}\cos x \end{bmatrix}$$

Eigenvalue  $-3$  and  $-39$ .

**Table IV:** Result of problem 3

$h$	Error in FO4PI	Error in SO4PI	Error in SO5PI	Error in SEO5PI
0.2	1.1547e+00	4.1748e-01	8.8252e+09	1.8230e+00
	6.0729e-01	2.1522e-01	8.3567e+09	8.9537e-01
0.4	3.2460e-01	5.9008e-02	7.1434e+19	1.4561e+00
	1.3734e-01	6.0322e-02	6.7641e+19	7.0320e-01
0.6	3.0740e-02	4.9960e-01	5.7820e+29	7.6093e-01
	7.0110e-02	3.4988e-01	5.4750e+29	2.9993e-01
0.8	1.7291e-01	1.0345e-01	4.6800e+39	9.0832e-01
	1.5614e-01	1.2152e-01	4.4316e+39	5.3685e-01
1.0	3.2241e-02	2.5627e-01	3.7881e+49	7.8614e-01
	9.6734e-03	1.4872e-01	3.5870e+49	3.9743e-01

This problem is solved by considering  $N = 100$  with  $tol = 10^{(-19)}$ . The results as presented in Table IV show that, the integrator SO5PI clearly perform better than the other three integrators.

**4. Conclusion**

Four- and five-point integrators were developed by considering three and four hybrid points which was determined using standard collocation method, and the adoption of collocation and interpolation technique. The integrators (2.9) and (2.20) were developed by varying the collocation points for the first derivative and keeping the collocation points for the second derivative constant. The integrators (2.14) and (2.26) were developed by keeping the collocation points for

the first derivative constant and varying the collocation points for the second derivative. The block integrators were also represented more compactly in Butcher tableau, the stability properties showed that the block integrators are stable, convergent and of order [5, 6, 6, 7] with large regions of absolute stability as shown in Fig. (1) – Fig. (4). It shows a better region of stability in the methods when varying the collocation points for the second derivative than those of keeping the collocation points for the first derivative constant.

The integrators were applied in solving test problems by writing a programme using MATLAB 8.5. The results of the numerical experiments in section three show that, developing integrators by varying the collocation points for the first derivative and keeping the collocation points for the second derivative constant at  $x_{n+1}$  leads to more efficient and accurate integrator than keeping the collocation points for the first derivative constant at  $0(\frac{1}{4})1$ ,  $0(\frac{1}{5})1$  and varying the collocation points for the second derivative. The order of the integrators increases as the hybrid points increases and the higher order integrators give better approximation with three hybrid points than with four hybrid points. The results of the new integrators where compared with the result of [1] and [3], and it is clear that the new integrators perform better especially SO5PI. Hence, the integrator SO5PI gives better approximation with the largest region of absolute stability when compared with the other three integrators developed, therefore, SO5PI is more suitable for the solution of stiff IVPs of ODEs.

**Conflict of interest**

The authors declare no conflict of interest.

**References**

- [1] H. Musa, M. B. Suleiman, and N. Senu, Fully Implicit 3-Point Block Extended Backward Differentiation Formula for stiff Initial Value Problems. *Appl. Math. Sci.*, 6(85), 4211-4228, 2012.

- [2] A. O. Adesanya, T. A. Anake, and G. J. Oghoyon, Continuous Implicit Method for the general second order ordinary differential equations. *J. of NAMP*, 15(November), 71-78, 2009
- [3] S. Adamu, K. Bitrus, and H. L. Buhari, One Step Second Derivative Method using Chebyshev Collocation Point for the solution of Ordinary Differential Equations. *J. of NAMP*, 51(May), 47-54, 2019.
- [4] S. D. Conte, and C. D. Boor, *Elementary Numerical Analysis: An Algorithmic Approach*. McGraw-Hill Book Company, USA, 1989.
- [5] D. G. Yakubu, and A. M. Kwami, Implicit two-derivative Runge-Kutta collocation methods for systems of initial value problems. *ScienceDirect*, *Journal of Nigerian Mathematical Society*, 34(2015), 128-142, 2015.
- [6] M. Z. M. Zabidi, Z. A. Majid, and N. Senu, Solving Stiff Differential Equations Using A-stable Block method. *International journal of Pure and Applied Mathematics*, 93 (3), 409-425, 2014.
- [7] H. M. Radzi, Z. Abdulmajid, F. Ismail and M. Suleiman, Two and Three Point One-Step Block Methods for Solving Delay Differential Equations. *Journal of Quality Measurement and Analysis*, 8(1), 29-41, 2012.
- [8] Z. A. Majid, M. S. Suleiman, M. F. Ismail and M. Othman, 2-Point Implicit Block One-Step Method Half Gauss-Seidel for Solving First Order Ordinary Differential Equations. *Matematika*, 19(2), 91-100, 2003.