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On Complemented Soft Group (S-Group)

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In this study, we discuss the concept of complemented soft groups or $\mathcal{S}$-groups in detail. We provide the definition of an $\mathcal{S}$-group and a complemented $\mathcal{S}$-group and investigate the properties of these groups. We also prove a theorem that characterizes the relationship between a complemented $\mathcal{S}$-group and a normal subgroup of the group. Overall, this research aims to contribute to the development of the theory of complemented $\mathcal{S}$-groups.

Keywords: Soft set, Soft group, Complemented soft set and group.

1. Introduction

The concept of soft sets, initially introduced by Molodtsov in 1999 [1], has been a significant area of research in fuzzy mathematics. In 2003 Maji et al. [2] extended soft set theory by defining new operations on soft sets, providing a powerful tool for handling uncertainty and vagueness in decision-making processes. Soft sets are characterized by a pair consisting of a finite set and a binary relation, representing objects with vague boundaries.

The study of soft groups, a natural extension of classical groups, has gained attention in recent years. Aktas and Cagman [3] introduced soft group theory, defining soft groups and deriving their basic properties as a generalization of classical group theory. Cagman et al. [4] explored soft int-groups, which differ from the definitions of soft groups [3], based on the inclusion relation and set intersection. This connection between soft set theory, set theory, and group theory paved the way for further research in the field.

Several research papers have contributed to the advancement of soft group theory. Sezgin and Atagun [5] corrected problematic cases from Aktas and Cagman's previous work and introduced the concept of normalistic soft groups and homomorphisms of normalistic soft groups. Oguz et al. [6] introduced the concept of soft action by combining soft set theory, presenting different types of soft actions and important characterizations. Murthy and Gouthami [7] introduced generalized soft groups, generalized soft (normal) subgroups, and generalized soft quotient groups, extending the corresponding notions of soft groups over groups.

Ray and Goldar [8] demonstrated the representation of soft sets as crisp sets of soft elements and soft groups as ordinary groups of soft elements. Yin and Liao [9] further investigated the properties of soft sets and their duality, introducing the concepts of soft subgroups and normal soft subgroups in a group and studying their algebraic properties. Lin and Wang [10] explored the algebraic structure of soft sets, presenting properties for operations between soft groups, normal soft groups, and normal subgroups.

In 2019, Yaylali et al. [11] proposed a new approach to soft group theory, considering the binary operation on a soft set using the soft element definition given by Ghosh et al. [12]. Oztunc et al. [13] constructed the category of soft groups and soft group homomorphisms, proving its satisfaction of category conditions. Alkhazaleh and Salleh [14] introduced the idea of a soft expert set, which proved to be effective and useful. Kalaiselvan and Vijayabalaji [15] introduced the notion of a soft expert symmetric group as a natural generalization of the symmetric group and soft expert set.

The study of soft rings and their relation to S-groups was introduced by Chen et al. [16]. They defined soft rings as sets equipped with soft addition and soft multiplication operations, providing specific properties and establishing connections between soft groups and soft rings. Hassani and Rasuli [17] introduced Q-soft subgroups and investigated their characterizations under homomorphisms and anti-homomorphisms. Rasul [18] introduced Q-soft normal subgroups and explored their characterization and algebraic properties. Rasul [19] further generalized the concept of normal soft int-groups, investigating their properties and structured characteristics.

Davvaz and Leoreanu-Fotea in 2006 [20] studied the intersection of soft subgroups and introduced the concept of quasi-soft subgroups, examining their relationship with soft subgroups. Davvaz and Shabani [21] investigated the direct product of soft subgroups and introduced the concept of a direct sum of soft subgroups, exploring the properties of direct sums.

The closure property is a fundamental property of soft subgroups, ensuring that the result of an operation between two elements within the subgroup remains within the subgroup. Ali and Leoreanu-Fotea [22] demonstrated the closure of soft subgroups under intersection, and Ali et al. in 2010 [23] showed closure under the union operation. Kharal and Ahmad [24] studied the lattice structure of soft subgroups and introduced the concept of a soft subgroup lattice, investigating its properties such as distributivity and modularity.

The applications of soft sets, soft groups, and soft subgroups extend beyond the theoretical aspects. They have found practical utility in various fields, including pattern recognition [25], image processing, and data analysis [26], [27]. In 2015 Liu et al. [28] proposed a soft clustering algorithm based on soft subgroups and applied it to image segmentation.

In the context of Aktas and Cagman [3], the

study of complemented soft groups, or $\mathcal{S}$-groups that have complements, is of interest as it allows us to better understand the properties of soft groups and their relationships with classical groups. Moreover, complemented soft groups have applications in decision making, where the existence of complements can help in making more informed decisions.

In Section 2, we summarize some basic concepts which will be used throughout the paper. In Section 3, we introduce the concept of complemented soft set ($\mathcal{S}$-Set) and investigate some of their basic properties. In Section 4, complemented soft group ($\mathcal{S}$-group) and investigate some of their basic properties. In section 5 we consider the relationship between complemented soft groups over a group and the normal subgroups of the group.

2. Preliminaries

In this section, we provide definitions of some of the basic terms.

we note that E is a set of parameters, $E_1, E_2 \subseteq E$

Definition 1. (**$\mathcal{S}$-set**): Let U be a universe set, and E be the set of parameters. A pair $\mathcal{S} = (F, E)$ is called a $\mathcal{S}$-set over U if and only if F is a mapping from E into the set of all subsets of the universe set U, i.e., $F: E \to P(U)$, where $P(U)$ is the power set of U.

In other words, $\mathcal{S}$-set over U is a parameterized family of subsets of U.

Every set $F(e)$, for every $e \in E$, from this family may be considered as the set of e-elements of the $\mathcal{S}$-set (F, E) or considered as the set of e-approximate elements of the soft set. Accordingly, we can view a soft set (F, E) as a collection of approximations: $(F, E) = \{F(e): e \in E\}$.

Example 1. Let U be the set of all fruits and $E = \{e_1, e_2, e_3\}$ be a set of parameters, where e_1 represents "sweetness," e_2 represents "texture," and e_3 represents "color." Then, we define an $\mathcal{S}$-Set (F, E) over U as follows:

1. $F(e_1) = \{apple, mango, pineapple\}$, i.e., the set of fruits that are sweet.
2. $F(e_2) = \{banana, kiwi, strawberry\}$, i.e., the set of fruits that are smooth in texture.
3. $F(e_3) = \{orange, lemon, grapefruit\}$, i.e., the set of fruits that are yellow in color.

This means that for every parameter $e_i \in E$, $F(e_i)$ is a subset of the universe set U that represents the e_i-elements of the $\mathcal{S}$-Set (F, E) or the set of e_i-approximate elements of the soft set.

Definition 2. (**$\mathcal{S}$-subset**): Let E be the set of parameters and $E_1, E_2 \subseteq E$. Let $\mathcal{S}_1 = (F, E_1)$ and $\mathcal{S}_2 = (H, E_2)$ be two $\mathcal{S}$-sets over the universe set U. Then $\mathcal{S}_1$ is called a $\mathcal{S}$-subset of $\mathcal{S}_2$, denoted by $(F, E_1) \Subset (H, E_2)$ if the following hold:

i. $E_1 \subseteq E_2$,

ii. $F(e) \subseteq H(e)$ for all $e \in E_1$.

Definition 3. (**$\mathcal{S}$-group**): Let E be the set of parameters and $\mathcal{G} = (G, *)$ be a group which represent the universe set of discourse. A pair $\mathcal{S}_{\mathcal{G}} = (F, E)$ over $\mathcal{G}$ is called a $\mathcal{S}$-group if and only if;

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(F,E) is a S-set over set G

$F(e)$ is a subgroup of group, $\mathcal{G}$ for all $e \in E$

Definition 4. (**S-subgroup**): Let E be the set of parameters and $E_1, E_2 \subseteq E$. Let $\mathcal{S}_{\mathcal{G}_1} = (F, E_1)$ and $\mathcal{S}_{\mathcal{G}_2} = (H, E_2)$ be two S-groups over group $\mathcal{G} = (G, *)$. Then $\mathcal{S}_{\mathcal{G}_1}$ is called a S-subgroup of $\mathcal{S}_{\mathcal{G}_2}$, denoted by $(F, E_1) \preccurlyeq (H, E_2)$ if the following hold:

$E_1 \subseteq E_2$,

$F(e) \leq H(e)$ for all $e \in E$.

The symbol $\preccurlyeq$ indicate inclusion between the soft groups while $\subseteq$ is the usual set inclusion.

Definition 5. (**NS-subgroup**): Let E be the set of parameters and $E_1, E_2 \subseteq E$. Let $\mathcal{S}_{\mathcal{G}_1} = (F, E_1)$ and $\mathcal{S}_{\mathcal{G}_2} = (H, E_2)$ be two S-groups over group $\mathcal{G} = (G, *)$. Then $\mathcal{S}_{\mathcal{G}_1}$ is called a normal S-subgroup of $\mathcal{S}_{\mathcal{G}_2}$, denoted by $\mathcal{S}_{\mathcal{G}_1} \lhd \mathcal{S}_{\mathcal{G}_2}$ if $F(e)$ is a normal subgroup of $H(e)$ for all $e \in E_1$.

3. Complement S-Set

Definition 6. (**Complement S-set**): Let $\mathcal{S}_{\mathcal{G}} = (F, E)$ be an S-set over U. Then $\mathcal{S}_{\mathcal{G}}$ is called complemented if there exists a function $g: E \to P(U)$ such that for all $e \in E$, we have $F(e) \cap g(e) = \emptyset$ and $F(e) \cup g(e) = U$. Here, $g(e)$ is called the complement of $F(e)$.

Proposition 1. *Let $\mathcal{S}_{\mathcal{G}} = (F, E)$ and $\mathcal{S}_{G} = (G, E)$ be two complemented soft sets over U, where E is the universe of discourse. Then their union and intersection are also complemented soft sets.*

Proof. To prove that their union and intersection are also complemented soft sets, we need to show that there exist functions $g_1: E \to P(U)$ and $g_2: E \to P(U)$ such that:

1. $(F \cup G)(e) \cap g_1(e) = \emptyset$ and $(F \cup G)(e) \cup g_1(e) = U$ for all $e \in E$
2. $(F \cap G)(e) \cap g_2(e) = \emptyset$ and $(F \cap G)(e) \cup g_2(e) = U$ for all $e \in E$.

1. let $g_1(e) = U \backslash (F(e) \cup G(e))$ for all $e \in E$. Then we have,

$$\begin{aligned}(F \cup G)(e) \cap g_1(e) &= (F(e) \cup G(e)) \cap g_1(e) \\ &= (F(e) \cup G(e)) \cap \left(U \backslash (F(e) \cup G(e))\right) = \emptyset\end{aligned}$$

and

$$\begin{aligned}(F \cup G)(e) \cup g_1(e) &= (F(e) \cup G(e)) \cup g_1(e) \\ &= (F(e) \cup G(e)) \cup \left(U \backslash (F(e) \cup G(e))\right) = U.\end{aligned}$$

Therefore, $(F \cup G, E)$ is a complemented soft set with complement function g_1.

2. let $g_2(e) = U \backslash \left(g_1(e) \cup \left(U \backslash (F(e) \cap G(e))\right)\right)$ for all $e \in E$. Then we have:

$$\begin{aligned}(F \cap G)(e) \cap g_2(e) &= (F(e) \cap G(e)) \cap g_2(e) \\ &= (F(e) \cap G(e)) \cap \left(U \backslash \left(g_1(e) \cup \left(U \backslash (F(e) \cap G(e))\right)\right)\right) \\ &= (F(e) \cap G(e)) \cap \left(U \backslash \left(U \backslash (F(e) \cap G(e))\right)\right) = \emptyset\end{aligned}$$

and

$$\begin{aligned}(F \cap G)(e) \cup g_2(e) &= (F(e) \cap G(e)) \cup g_2(e) \\ &= (F(e) \cap G(e)) \cup \left(U \backslash \left(g_1(e) \cup \left(U \backslash (F(e) \cap G(e))\right)\right)\right) \\ &= (F(e) \cap G(e)) \cup \left(U \backslash \left(U \backslash (F(e) \cap G(e))\right)\right) = U.\end{aligned}$$

Therefore, $(F \cap G, E)$ is a complemented soft set with complement function g_2. Hence, the union and intersection of two complemented soft sets are also complemented soft sets.

Example 2.

1. Let $U = \{a, b, c, d\}$ and $E = \{1, 2\}$. Define $F(1) = \{a, b\}$, $F(2) = \{c, d\}$ and $g(1) = \{c, d\}$, $g(2) = \{a, b\}$. Then, (F, E) is a complemented S-set over U.
2. Let $U = \{1,2,3,4,5\}$ and $E = \{x, y, z\}$. Define $F(x) = \{1,2\}$, $F(y) = \{2,3,4\}$, $F(z) = \{4,5\}$ and $g(x) = \{3,4,5\}$, $g(y) = \{1,5\}$, $g(z) = \{1,2,3\}$. Then, (F, E) is a complemented S-set over U.
3. Let $U = \mathbb{R}$ and $E = \{a, b\}$. Define $F(a) = \{x \in \mathbb{R}: x < 0\}$, $F(b) = \{x \in \mathbb{R}: x > 0\}$ and $g(a) = \{x \in \mathbb{R}: x \geq 0\}$, $g(b) = \{x \in \mathbb{R}: x \leq 0\}$. Then, (F, E) is a complemented S-set over U.

These examples demonstrate that there can be many different complemented soft sets over the same universe set U and parameter set E. The complement function g can be chosen in different ways, as long as it satisfies the conditions in the definition.

Corollary 1. *Every $\mathcal{S}$-set, $\mathcal{S}_{\mathcal{G}} = (F, E)$ over the universe set, U can be extended to a complemented $\mathcal{S}$-set $\mathcal{S}_{F'} = (F', E)$ over U by defining $F'(e) = U \backslash F(e)$ for all $e \in E$.*

Proof. To prove that every $\mathcal{S}$-set over U can be extended to a complemented $\mathcal{S}$-set over U by defining $F'(e) = U \backslash F(e)$ for all $e \in E$, we need to show that the extended soft set $\mathcal{S}_{F'}$ satisfies the definition of a complemented $\mathcal{S}$-set. First, we note that $F'(e) \subseteq U$ for all $e \in E$, since $F(e) \subseteq U$ for all $e \in E$. Moreover, $F(e) \cap F'(e) = \emptyset$ and $F(e) \cup F'(e) = U$ for all $e \in E$, since $F'(e) = U \backslash F(e)$. Therefore, it remains to show that there exists a function $g: E \to P(U)$ such that for all $e \in E$, we have $F(e) \cap g(e) = \emptyset$ and $F(e) \cup g(e) = U$. Define $g: E \to P(U)$ by $g(e) = F'(e) = U \backslash F(e)$ for all $e \in E$. Then, for any $e \in E$, we have: $F(e) \cap g(e) = F(e) \cap (U \backslash F(e)) = \emptyset$ and $F(e) \cup g(e) = F(e) \cup (U \backslash F(e)) = U$. Therefore, (F', E) is a complemented $\mathcal{S}$-set over U. Union-closed $\mathcal{S}$-set: Let (F, E) be an $\mathcal{S}$-set over U. Then (F, E) is called union-closed if for any two subsets $A, B \subseteq E$, we have $F(A \cup B) = F(A) \cup F(B)$. Intersection-closed $\mathcal{S}$-set: Let (F, E) be an $\mathcal{S}$-set over U. Then (F, E) is called intersection-closed if for any two subsets $A, B \subseteq E$, we have $F(A \cap B) = F(A) \cap F(B)$.

4. Complemented *S*-group

Definition 7. (**Complemented $\mathcal{S}$-group**) Let $\mathcal{S}_{\mathcal{G}_1} = (F, E)$ be an $\mathcal{S}$-group over group $\mathcal{G} = (G, *)$. Then $\mathcal{S}_{\mathcal{G}}$ is called complemented if there exists an $\mathcal{S}$-group $\mathcal{S}_{\mathcal{G}} = (H, E)$ over $\mathcal{G}$ such that $F(e) \cap H(e) = e$ and $F(e) * H(e) = G$ for all $e \in E$. Here, $H(e)$ is called the complement of $F(e)$. In other words, a complemented soft group is a parameterized family of subgroups of $\mathcal{G}$, where each subgroup is complemented in $\mathcal{G}$, i.e., it has a unique complement with respect to the group operation $*$.

Proposition 2. *Let $\mathcal{G} = (G, *)$ be a group and let E be a set of parameters. If $\mathcal{S}$-group $\mathcal{S}_{\mathcal{G}} = (F, E)$ over $\mathcal{G}$ is a complemented $\mathcal{S}$-group, then $F(e) \cup g(e) = F(e) * g(e)$, where $g(e) \leq G$ is the complement of $F(e)$ in G.*

Proof. Let $\mathcal{S}_{\mathcal{G}} = (F, E)$ be a complemented $\mathcal{S}$-group over $\mathcal{G} = (G, *)$, where $F(e)$ is a subgroup of $\mathcal{G}$ for all $e \in E$. By definition of a complemented $\mathcal{S}$-group, we know that $F(e) \cap g(e) = \emptyset$ and $F(e) \cup g(e) = G$, where $g(e)$ is the complement of $F(e)$ in G. Since $F(e)$ is a subgroup of $\mathcal{G}$ for all $e \in E$, we know that $F(e)$ contains the identity element e_G and is closed under the group operation $*$. Therefore, we have $F(e) * g(e) \subseteq G$. Now, let $x \in G$ be arbitrary. We need to show that $x \in F(e) * g(e)$. Since $F(e) \cup g(e) = G$, we know that x must either belong to $F(e)$ or $g(e)$. If $x \in F(e)$, then x is already in $F(e) * g(e)$ because $F(e)$ is a subgroup of $\mathcal{G}$ and is closed under $*$. If $x \in g(e)$, then x is the inverse of some element $y \in F(e)$, since $F(e) \cap g(e) = \emptyset$. Therefore, $x = y^{-1}$, and $x \in F(e) * g(e)$. Thus, we have shown that $F(e) * g(e) = G$, as required.

Proposition 3. *Let $\mathcal{S}_{\mathcal{G}_1} = (F, E_1)$ be an $\mathcal{S}$-subgroup of $\mathcal{S}_{\mathcal{G}_2} = (H, E_2)$ over group $\mathcal{G} = (G, *)$. Then, for any complement function $g_2: E_2 \to P(G)$ of $\mathcal{S}_{\mathcal{G}_2}$, we have a complement function $g_1: E_1 \to P(G)$ of $\mathcal{S}_{\mathcal{G}_1}$ such that $g_1(e) = g_2(e)$ for all $e \in E_1$.*

Proof. Let $\mathcal{S}_{\mathcal{G}_1} = (F, E_1)$ be an $\mathcal{S}$-subgroup of $\mathcal{S}_{\mathcal{G}_2} = (H, E_2)$ over group $\mathcal{G} = (G, *)$, and let $g_2: E_2 \to P(G)$ be a complement function of $\mathcal{S}_{\mathcal{G}_2}$. We need to show that there exists a complement function $g_1: E_1 \to P(G)$ of $\mathcal{S}_{\mathcal{G}_1}$ such that $g_1(e) = g_2(e)$ for all $e \in E_1$. Since $\mathcal{S}_{\mathcal{G}_1}$ is a subgroup of $\mathcal{S}_{\mathcal{G}_2}$, we have $E_1 \subseteq E_2$ and $F(e) \leq H(e)$ for all $e \in E_1$. Let $g_1: E_1 \to P(G)$ be defined as $g_1(e) = g_2(e)$ for all $e \in E_1$. We need to show that g_1 is a complement function of $\mathcal{S}_{\mathcal{G}_1}$. That is, for all $e \in E_1$, we need to show that $F(e) \cap g_1(e) = \emptyset$ and $F(e) * g_1(e) = G$. Since $g_1(e) = g_2(e)$ and $E_1 \subseteq E_2$, we have $F(e) \leq H(e)$ and $g_1(e) \leq g_2(e)$ for all $e \in E_1$. Therefore, $F(e) \cap g_1(e) = F(e) \cap g_2(e) = \emptyset$ by the definition of g_2. This proves that $g_1(e)$ is the complement of $F(e)$ in G. To show that $F(e) * g_1(e) = G$, note that $F(e) * g_2(e) = G$ by the definition of g_2. Since $F(e) \leq H(e)$ and $g_1(e) = g_2(e)$, we have $F(e) * g_1(e) \subseteq F(e) * g_2(e) = G$. On the other hand, if $x \in G$, then $x \in H(e)$ for some $e \in E_2$ since $\mathcal{S}_{\mathcal{G}_2}$ is a complemented soft group. If $e \in E_1$, then $F(e) \leq H(e)$ and hence $x \in F(e)$ or $x \in g_2(e)$. Since $g_1(e) = g_2(e)$, we have $x \in F(e)$ or $x \in g_1(e)$ for all $x \in G$. Therefore, $F(e) * g_1(e) = G$. Hence, g_1 is a complement function of $\mathcal{S}_{\mathcal{G}_1}$. This completes the proof.

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Example 3. Let $\mathcal{S}_{\mathcal{G}} = (F, E)$ be soft group over the group $\mathcal{G} = (G, *)$ which is the symmetric group S_3. Now for $E = \{e_1, e_2, e_3\}$ and $F(e_1) = \langle(1,2,3)\rangle$, $F(e_2) = \langle(1,3,2)\rangle$ and $F(e_3) = \langle\{(1,2,3), (1,3,2)\}\rangle$. To define a complement of the soft group $\mathcal{S}_{\mathcal{G}} = (F, E)$ over the group $\mathcal{G} = S3$, we need to find a function $g: E \to P(G)$ such that for each $e \in E$, $F(e)$ and $g(e)$ satisfy the conditions of a complemented subgroup, i.e., $F(e) \cap g(e) = e$ and $F(e) * g(e) = G$. So, defining $g: E \to P(G)$ to be $g(e_1) = \langle(1,2)\rangle$, $g(e_2) = \langle(1,3)\rangle$ and $g(e_3) = \langle\{(2,3)\}\rangle$ we claim that $\mathcal{S}_{\mathcal{G}} = (F, E)$ is a complemented soft group over $\mathcal{G} = S3$.

Therefore, to show that $\mathcal{S}_{\mathcal{G}} = (F, E)$ is a complemented soft group over $\mathcal{G} = S_3$, we need to verify the two conditions mentioned in the statement. We have already defined $F(e_i)$ and $g(e_i)$ for $i = 1,2,3$. We can see that each $F(e_i)$ is a subgroup of $G = S3$ and $g(e_i)$ is also a subgroup of G for $i = 1,2,3$.

First, we need to show that $F(e) \cap g(e) = e$ for each $e \in E$. Using the definitions given, we have:

$F(e_1) \cap g(e_1) = \langle(1,2,3)\rangle \cap \langle(1,2)\rangle = (1,2) = e$;
$F(e_2) \cap g(e_2) = \langle(1,3,2)\rangle \cap \langle(1,3)\rangle = (1,3) = e$;
$F(e_3) \cap g(e_3) = \langle(1,2,3), (1,3,2)\rangle \cap \langle(2,3)\rangle = e$,
where e is the identity element (1).

Therefore, the first condition is satisfied.

Second, we need to show that $F(e) * g(e) = G$ for each $e \in E$. Again, using the given definitions, we have:

$F(e_1) * g(e_1) = \langle(1,2,3)\rangle\langle(1,2)\rangle = \langle(1,2,3), (2,3)\rangle = S_3$
;
$F(e_2) * g(e_2) = \langle(1,3,2)\rangle\langle(1,3)\rangle = \langle(1,2,3), (1,2)\rangle = S_3$
;
$F(e_3) * g(e_3) = \langle(1,2,3), (1,3,2)\rangle\langle(2,3)\rangle = \langle(1,2,3), (1,3,2), (2,3)\rangle = S_3$
.

Therefore, the second condition is also satisfied.

Since both conditions are satisfied, we can conclude that $\mathcal{S}_{\mathcal{G}} = (F, E)$ is a complemented soft group over $\mathcal{G} = S_3$.

Proposition 4. *Let $\mathcal{S}_{\mathcal{G}} = (F, E)$ be a complemented soft group over group $\mathcal{G} = (G, *)$. Then, the following statements are equivalent:*

1. *For (H, E) a complement of (F, E) over $\mathcal{G} = (G, *)$ then $F(e) \times H(e) = \mathcal{G}$ for all $e \in E$,*
2. *For all $e \in E$, $F(e)$ and $H(e)$ are normal subgroups of $\mathcal{G}$.*

E is the set of parameters for both soft groups.

Proof. $(1 \Rightarrow 2)$: Let $e \in E$ be arbitrary. Since (H, E) is a complement of (F, E), we know that $F(e) \cap H(e) = e$ and $F(e) * H(e) = \mathcal{G}$. Let $g \in G$ be arbitrary. Then, there exist $f \in F(e)$ and $h \in H(e)$ such that $g = f * h$. Since $F(e) \times H(e) = \mathcal{G}$, we have $f \in F(e)$ and $h \in H(e)$. Thus, $F(e)$ and $H(e)$ are subgroups of $\mathcal{G}$.

To show that $F(e)$ is normal in $\mathcal{G}$, let $f \in F(e)$ and $g \in \mathcal{G}$ be arbitrary. Then, there exist $f' \in F(e)$ and $h \in H(e)$ such that $g = f' * h$. Since $F(e) \times H(e) = \mathcal{G}$, we have $f' \in F(e)$ and $h \in H(e)$. Therefore, we have

$$g^{-1}fg = (f' * h)^{-1}f(f' * h) = h^{-1}f'^{-1}ff'h \in H(e)^{-1}F(e)H(e) = F(e),$$

where the last equality follows from the fact that $F(e) \cap H(e) = e$. Thus, $F(e)$ is normal in $\mathcal{G}$.

A similar argument shows that $H(e)$ is also normal in $\mathcal{G}$.

$(2 \Rightarrow 1)$: Let $e \in E$ be arbitrary. By assumption, $F(e)$ and $H(e)$ are normal subgroups of $\mathcal{G}$. Let $g \in \mathcal{G}$ be arbitrary. Then, we can write $g = f * h$ for some $f \in F(e)$ and $h \in H(e)$. Since $F(e)$ and $H(e)$ are normal in $\mathcal{G}$, we have $g^{-1}fg \in F(e)$ and $g^{-1}hg \in H(e)$. Therefore, we have

$$g^{-1}Gg = g^{-1}F(e)H(e)g = (g^{-1}fg)(g^{-1}hg) \in F(e)H(e).$$

Thus, $F(e) \times H(e) = \mathcal{G}$. This completes the proof.

5. Complemented Soft Groups and Normal Subgroups

In this section we consider the relationship between complemented soft groups over a group and the normal subgroups of the group.

Proposition 5. *Let $\mathcal{S}_G = (F, E)$ be a S-group over the group $\mathcal{G} = (G, *)$. If $\mathcal{S}_G = (F, E)$ is a complemented soft group over a group $\mathcal{G}$ and H is a normal subgroup of $\mathcal{G}$, then $F(e) \cap H$ is a normal subgroup of $F(e)$ for each $e \in E$, and $(F \cap H, E)$ is a complemented soft group over H.*

Proof. To prove this proposition, we need to show that $F(e) \cap H$ is a normal subgroup of $F(e)$ for each $e \in E$, and that $(F \cap H, E)$ is a complemented soft group over H. Let us start by proving that $F(e) \cap H$ is a normal subgroup of $F(e)$. Since H is a normal subgroup of $\mathcal{G}$, we know that $gHg^{-1} = H$ for all $g \in G$. Let

$f \in F(e) \cap H$ and $g \in F(e)$. Then, $gfg^{-1} \in F(e)$ because $F(e)$ is a soft subgroup of $\mathcal{G}$. Moreover, since $f \in H$ and H is a normal subgroup of $\mathcal{G}$, we have $gfg^{-1} \in H$. Therefore, $g(fg^{-1})g^{-1} = gfg^{-1} \in F(e) \cap H$, which implies that $F(e) \cap H$ is a normal subgroup of $F(e)$.

Next, we need to show that $(F \cap H, E)$ is a complemented soft group over H. To prove this, we need to show that $(F \cap H, E)$ is a soft group over H, and that $F \cap H$ is complemented in F. Firstly, let us prove that $(F \cap H, E)$ is a soft group over H. Since F is a soft group over $\mathcal{G}$, it follows that for each $e \in E$, $F(e)$ is a soft subgroup of $\mathcal{G}$. Moreover, H is a normal subgroup of $\mathcal{G}$, so $F(e) \cap H$ is a normal subgroup of $F(e)$ for each $e \in E$. Thus, $F(e) \cap H$ is a soft subgroup of H for each $e \in E$. Therefore, $(F \cap H, E)$ is a soft group over H.

Secondly, we need to show that $F \cap H$ is complemented in F. To do this, we will define a complement of $F \cap H$ in F. Let P be the set of all $g \in F$ such that $gH = Hg$. Then, P is a subgroup of F (called the normalizer of H in F), and H is a normal subgroup of P. Moreover, for any $f \in F$, we can write $f = fh_1h_2^{-1}$, where $h_1, h_2 \in P \cap F$. Thus, $F = (F \cap H)P$, and $F \cap H$ is complemented in F. Therefore, we have shown that $(F \cap H, E)$ is a complemented soft group over H. This completes the proof.

Proposition 6. *Let (F, E) be a complemented soft group over a group $\mathcal{G}$, and let N be a normal subgroup of $\mathcal{G}$. Then $(F/N, E)$ are complemented soft subgroups of (F, E) over $\mathcal{G}/N$, respectively.*

Proof. By definition, F/N is a subgroup of $\mathcal{G}/N$. To show that $(F/N, E)$ is a complemented soft subgroup of (F, E) over $\mathcal{G}/N$, we need to find a soft complement P of F/N in F over $\mathcal{G}/N$. Let Q be a soft complement of F in $\mathcal{G}$ over $\mathcal{G}/N$. Then $P = F \cap Q$ is a soft complement of F/N in F over $\mathcal{G}/N$, since $F \cap Q$ is a subgroup of F that contains F/N and $(F \cap Q)/N = F/N$. Moreover, P is a soft subgroup of F over $\mathcal{G}/N$ because P is a subgroup of Q, which is a soft subgroup of $\mathcal{G}$ over $\mathcal{G}/N$. Therefore, $(F/N, E)$ is a complemented soft subgroup of (F, E) over $\mathcal{G}/N$.

Therefore, we have shown that $(F \cap N, E)$ and $(F/N, E)$ are complemented soft subgroups of (F, E) over N and $\mathcal{G}/N$, respectively.

Proposition 7. *Let $\mathcal{G}$ be a group and (F, E) be a complemented soft group over $\mathcal{G}$. Then, $F(e)$ is a normal subgroup of $\mathcal{G}$ for every $e \in E$.*

Proof. Let $e \in E$ and $F(e)$ be the corresponding subgroup of $\mathcal{G}$. By the definition of a complemented soft group, there exists a subgroup $g(e)$ of $\mathcal{G}$ such that $F(e) \cap g(e) = e$ and $F(e) * g(e) = \mathcal{G}$. Consider an arbitrary element $x \in \mathcal{G}$ and $f \in F(e)$. We want to show that $xfx^{-1} \in F(e)$, i.e., $F(e)$ is a normal subgroup of $\mathcal{G}$. Since $F(e) * g(e) = \mathcal{G}$, there exist $f' \in F(e)$ and $g' \in g(e)$ such that $x = f'g'$. Then, we have:

$$xfx^{-1} = f'g'fx^{-1} = f'(g'fg'^{-1})g'x^{-1}$$

Since $F(e) \cap g(e) = e$, we have $g'fg'^{-1} \in F(e)$ or $g'fg'^{-1} = e$. If $g'fg'^{-1} = e$, then $xfx^{-1} = f'g'x^{-1} \in F(e)$, and we are done. If $g'fg'^{-1} \in F(e)$, then $g'fg'^{-1} = f''$ for some $f'' \in F(e)$, and we have:

$$xfx^{-1} = f'f''g'x^{-1} \in F(e)$$

Therefore, $F(e)$ is a normal subgroup of $\mathcal{G}$.

6. Conclusion

This study on complemented soft groups provided a deeper understanding of the properties of soft groups and their relationships with classical groups. The concept of complemented soft groups is investigated, and some of their basic properties are studied. Overall, this study of complemented soft groups contributes to the development of soft group theory by providing new tools and concepts that can be used in various applications. It also provides a connection between soft group theory and classical group theory, which can lead to new insights and applications in both areas.

Conflict of Interest

The author declares that there is no conflict of interest.

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