



Article Info

Received: 25th February 2023Revised: 20th October 2023Accepted: 22nd October 2023

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Cite this: *CaJoST*, 2023, 2, 154-162

Global stability and bifurcation analysis on the dynamics of three interacting populations in prey-predator system

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A mathematical model was proposed and analysed to study the dynamics of a system of a prey, primary predator and a super predator in which the predators show Holling Type II functional response to their respective prey. The study substantially provides mechanism for improving the survivability of honeybee population and to halt the negative impact of *Varroa-mite* infestation on honeybee population by introducing *Pseudoscorpion* to serve as a super predator of *Varroa-mite*. Also, the study significantly shows the impact of virus on honeybee infested with virus-carrying *Varroa-mite*. The global stability analysis and Hopf-bifurcation of the interior steady state existence of the system are studied and explored. The analytical results matched with the numerical results of the model developed.

Keywords: Global Stability, Bifurcation, Dynamics, Prey-predator and Interior Steady state.

1. Introduction

In a common natural setting of ecosystem, where species interact in their habitat is ideally supported by a food chain. The relationship among species could be those of predator-prey, mutualism, competition for resources in the environment they occupy [1]. The pertinent question ask by all is that, if the food chain is allowed to be exploited unchecked in their natural habitat, what become of the ecosystem? This has been the energy expended by proponents of ecosystem and researchers in attempting to solve in the field of ecology and mathematical biology. [2] and [3] are the simple and fundamental model that shed lights on the prey-predator system. Afterwards, many more mathematical models on prey-predator were built in a bit to explain complex nature of certain phenomena in natural occurrence; such models includes: [4], [5], [6], [7], [8] and among others.

In a sophisticated way, [9] proposed a three ecological models in multi-species interactions with the existence of a reserved area, so as to prevent the slow extinction of some endangered species and to test the stability when the length of the food chain and size of the web models are increased. Also, [10] developed a mathematical model that explains two essential ecosystem services, known as resilience and yield, in a harvested two predators, one prey system. The study investigated how these two ecosystem

services respond to various harvesting plans applied to the structure of the interactions. In individual harvesting, we observe that prey harvesting is suitable for generating more yield while either predator harvesting gives more resilience. Several patterns are possible based on the intensity of harvesting efforts applied to prey or the predator in simultaneous harvesting. The study reveals that a balanced harvest between prey and predators may give more than the selective harvesting of any species yielding and stabilizing the ecosystem.

The importance of honeybees in sustaining ecosystems, regional, national and global food supplies, and various agricultural industries cannot be underestimated [11]. But the population of honeybee invariably declines due to many stressors more especially *Varroa-mite* infestation. In the manner of demonstrating the natural interaction between honeybee and *Varroa-mite*, researches were developed and studied. In that light, [12] studied a prey-predator system that highlighted an infested parasitic *Varroa-mite* and honeybee natural interaction. The study reveals high negative impacts of *Varroa-mite* infestation on honeybee coupled with virus infection. Thus, honeybee population declines sharply due to *Varroa-mite* infestation and virus infection and eventually resulted to the honeybee colony collapse. In an attempt to

remedying the menace, mathematical models that were developed includes: [13], [14], [15], [16] and among others. In all models hitherto studied and reviewed to the best of my knowledge, none of them had proposed a model that describe honeybee and *Varroa-mite* interaction in prey-predator system.

In this study, the main central focus was to develop a mathematical model that represent an interplay between honeybee and *Varroa-mite* in prey-predator system in the presence of *Pseudoscorpion* serving as a super predator of *Varroa-mite*. This natural interplay between these three species has gone a long way to stabilize the ecosystem, unlike the interplay between honeybee and *Varroa-mite* only.

2. Model Formulation

The honeybee, *Varroa-mite* and *Pseudoscorpion* population are denoted as $H(t)$, $V(t)$ and $P(t)$ respectively. It is assumed that honeybee population grows in logistic manner. It is assumed that *Varroa-mite* carrying virus infested (invaded) healthy honeybee only. *Varroa-mite* is a primary predator of honeybee, while, *Pseudoscorpion* is a super predator of *Varroa-mite*. All populations of these species are consider to die naturally. It is also assumed that, virus carried by *Varroa-mite* doesn't harm the super predator (*Pseudoscorpion*). Furthermore, it is assumed that, there is no competition among the super predators neither on the primary predators. Hence, the population of the honeybee reproduces in logistic manner by the term $\gamma H \left(1 - \frac{H}{k}\right)$, where γ is the intrinsic

growth, k is the environmental carrying capacity of honeybee. It decline by a predation term inform of Holing Type II functional response $\frac{\alpha_1 HV}{1 + b_1 H}$, where α_1 is an infestation parameter of *Varroa-mite* on honeybee, b_1 saturation rate of honeybee. It also decline by natural death μ_1 and death due to virus invasion δ . The population of *Varroa-mite* grows by the term $\eta_1 \frac{\alpha_1 HV}{1 + b_1 H}$, where η_1 is conversion coefficient of *Varroa-mite* on honeybee. It declines by the predation rate of *Pseudoscorpion* $\frac{\alpha_2 HV}{1 + b_2 H}$, where α_2 is a predation(capture) parameter of *Pseudoscorpion* on *Varroa-mite*, b_2

saturation rate of *Varroa-mite*. It also decline by natural death at rate μ_2 . Similarly, the population of *Pseudoscorpion* grows by term $\eta_2 \frac{\alpha_2 HV}{1 + b_2 H}$ where η_2 is conversion coefficient of predation of *Pseudoscorpion* on *Varroa-mite* and declines by natural death at rate μ_3 .

Therefore, with the assumptions, state variables and parameters hitherto stated, give the dynamical system of super predator, primary predator and a single prey. The prey is contracted with disease by factor of *Varroa-mite* being a vector of certain viruses. Thus, the dynamical system gives the followings:

$$\left. \begin{aligned} \frac{dH}{dt} &= \gamma H \left(1 - \frac{H}{k}\right) - \frac{\alpha_1 HV}{1 + b_1 H} - \mu_1 H - \delta H \\ \frac{dV}{dt} &= \eta_1 \frac{\alpha_1 HV}{1 + b_1 H} - \mu_2 V - \frac{\alpha_2 VP}{1 + b_2 V} \\ \frac{dP}{dt} &= \eta_2 \frac{\alpha_2 VP}{1 + b_2 V} - \mu_3 P \\ H(0) &\geq 0, V(0) \geq 0, P(0) \geq 0 \end{aligned} \right\} \quad (1)$$

3. Basic Property of the Model

3.1. Positivity of the model

Theorem 1: Let the initial data for the model (1) be $H(0) \geq 0$, $V(0) \geq 0$ and $P(0) \geq 0$. Then the solution (H, V, P) of the model equation (1) with positive initial data, will remain positive for all time $t > 0$.

Proof:

Let $t_1 = \text{Sup}\{t > 0 : H \geq 0, V \geq 0, P \geq 0 \in [0, t]\}$,

thus $t > 0$. Then, it follows from the equations of the model (1), we have the followings:

From the first equation of model (1), we have:

$$\frac{dH}{dt} \geq -\frac{\alpha_1 HV}{1 + b_1 H} - \mu_1 H - \delta H \geq -\left[\frac{\alpha_1 V}{1 + b_1 H} - \mu_1 - \delta\right] H$$

Using separation of variables method, we have:

$$\frac{dH}{H} \geq -\left[\frac{\alpha_1 V}{1 + b_1 H} - \mu_1 - \delta\right] dt$$

Integrating both sides and taking anti-log, we have:

$$\ln H \geq -\int_0^{t_1} \left(\frac{\alpha_1 V(y)}{1 + b_1 H}\right) dy - \int (\mu_1 + \delta) dt$$

$$\text{So that, } H(t) \geq e^{-\int_0^{t_1} \left(\frac{\alpha_1 V(y)}{1 + b_1 H}\right) dy - (\mu_1 + \delta)t}$$

At $t_1 = 0$ and with initial condition $H(0)$
Then,

$$H(t) \geq H(0)e^{-\int_0^t \left(\frac{\alpha_1 V(y)}{1+b_1 H} \right) dy - (\mu_1 + \delta)t} > 0,$$

since $\left(\frac{\alpha_1 V(y)}{1+b_1 H} \right) > 0$ and $(\mu_1 + \delta) > 0$ (2)

From the second equation of model (1), we have:

$$\frac{dV}{dt} \geq -\frac{\alpha_2 VP}{1+b_2 V} - \mu_2 V \geq -\left[\frac{\alpha_2 P}{1+b_2 V} - \mu_2 \right] V$$

Using separation of variables method, we have:

$$\frac{dV}{V} \geq -\left[\frac{\alpha_2 P}{1+b_2 V} - \mu_2 \right] dt$$

Integrating both sides and taking anti-log, we have:

$$\ln V \geq -\int_0^t \left(\frac{\alpha_2 P(y)}{1+b_2 V} \right) dy - \int_0^t \mu_2 dt$$

$$\text{So that, } V(t) \geq e^{-\int_0^t \left(\frac{\alpha_2 P(y)}{1+b_2 V} \right) dy - \mu_2 t}$$

At $t_1 = 0$ and with initial condition $V(0)$

Then,

$$V(t) \geq V(0)e^{-\int_0^t \left(\frac{\alpha_2 P(y)}{1+b_2 V} \right) dy - \mu_2 t} > 0,$$

$$\text{since } \left(\frac{\alpha_2 P(y)}{1+b_2 V} \right) > 0 \text{ and } \mu_2 > 0$$

(3)

From the third equation of model (1), we have:

$$\frac{dP}{dt} \geq -\mu_3 P$$

Using separation of variables method, we have:

$$\frac{dP}{P} \geq -\mu_3 dt$$

Integrating both sides and taking anti-log, we have:

$$\ln P \geq -\int_0^t \mu_3 dt$$

$$\text{So that, } P(t) \geq e^{-\mu_3 t}$$

At $t = 0$ and with initial condition $P(0)$

Then,

$$P(t) \geq P(0)e^{-\mu_3 t} > 0, \text{ since } \mu_3 > 0 \quad (4)$$

From the argument so far presented in inequalities (2), (3) and (4), it can be shown that the state variables are all positive for all $t > 0$. Hence, the solution of model (1) remains positive for all $t > 0$.

4. Results and Discussion

In this section, the study established the global stability of the interior steady state only and existence of hopf bifurcation analysis alongside the numerical solution of model equations (1).

4.1 Global stability of the interior steady state $E_8(H^*, V^*, P^*)$ of the model

Global stability of the interior steady state $E_8(H^*, V^*, P^*)$, analyzing the global stability of the interior steady state, a suitable lyapunov function is chosen from which conditions for the global asymptotic stability of the interior steady state E_8 are derived. Therefore, the approach is motivated by [17] and [18] respectively.

Theorem 2: the interior steady state $E_8(H^*, V^*, P^*)$ is globally asymptotically stable if the following conditions hold:

- i. $\gamma > 0, \gamma > \mu_1$ and $\gamma > \delta$
- ii. $\gamma > \alpha_1$
- iii. $\alpha_1 > \mu_2, \alpha_1 > \alpha_2$ and $\alpha_2 > \mu_3$

Proof: Consider a lyapunov function

$$V(H, V, P) = (H - H^*) - H^* \log\left(\frac{H}{H^*}\right) \quad (5)$$

$$+ (V - V^*) - V^* \log\left(\frac{V}{V^*}\right) + (P - P^*) - P^* \log\left(\frac{P}{P^*}\right)$$

Simplifying the above lyapunov function, we have:

$$V(H, V, P) = H - H^* - (H^* \log H - H^* \log H^*) \quad (6)$$

$$+ V - V^* - (V^* \log V - V^* \log V^*) + P - P^* - (P^* \log P - P^* \log P^*)$$

$$V(H, V, P) = H - H^* - H^* \log H + H^* \log H^* \quad (7)$$

$$+ V - V^* - V^* \log V + V^* \log V^* + P - P^* - P^* \log P + P^* \log P^*$$

$$\text{Where: } \frac{\partial V}{\partial H} = 1 - \frac{H^*}{H} = \left(\frac{H - H^*}{H} \right),$$

$$\frac{\partial V}{\partial V} = 1 - \frac{V^*}{V} = \left(\frac{V - V^*}{V} \right) \text{ and } \frac{\partial V}{\partial P} = 1 - \frac{P^*}{P} = \left(\frac{P - P^*}{P} \right)$$

Differentiating with respect to time t

$$\frac{dV}{dt} = \frac{\partial V}{\partial H} \frac{dH}{dt} + \frac{\partial V}{\partial V} \frac{dV}{dt} + \frac{\partial V}{\partial P} \frac{dP}{dt}$$

$$\frac{dV}{dt} = \frac{H - H^*}{H} \left[\gamma H \left(1 - \frac{H}{k} \right) - \frac{\alpha_1 HV}{1+b_1 H} - \mu_1 H - \delta H \right]$$

$$+ \frac{V - V^*}{V} \left[\eta_1 \frac{\alpha_1 HV}{1+b_1 H} - \mu_2 V - \frac{\alpha_2 VP}{1+b_2 V} \right] + \frac{P - P^*}{P} \left[\eta_2 \frac{\alpha_2 VP}{1+b_2 V} - \mu_3 V \right]$$

$$\begin{aligned}
 &= (H - H^*) \left[\gamma \left(1 - \frac{H}{k} \right) - \frac{\alpha_1 V}{1 + b_1 H} - \mu_1 - \right] \\
 &+ (V - V^*) \left[\eta_1 \frac{\alpha_1 H}{1 + b_1 H} - \mu_2 - \frac{\alpha_2 P}{1 + b_2 V} \right] + (P - P^*) \left[\eta_2 \frac{\alpha_2 V}{1 + b_2 V} - \mu_2 \right] \\
 &= (H - H^*) \left[\gamma - \frac{\gamma H}{k} - \frac{\alpha_1 V}{1 + b_1 H} - \mu_1 - \delta - \gamma + \gamma + \frac{\alpha_1 V}{1 + b_1 H} + \mu_1 + \delta \right] + (V - V^*) \left[\eta_1 \frac{\alpha_1 H}{1 + b_1 H} - \mu_2 - \frac{\alpha_2 P}{1 + b_2 V} + \mu_2 \right] \\
 &\quad + (P - P^*) \left[\eta_2 \frac{\alpha_2 V}{1 + b_2 V} - \mu_2 - \eta_2 \frac{\alpha_2 V}{1 + b_2 V} + \mu_2 \right] \\
 &= (H - H^*) \left[\gamma (H - H^*) + \alpha_1 (V - V^*) \right] + (V - V^*) \\
 &\quad \left[\alpha_1 (V^2 - V^*) + \frac{\alpha_1 (H - H^*)}{1 + b_1 H^*} \right] + (P - P^*) \left[\frac{\alpha_2 (V - V^*)}{1 + b_1 V^*} \right]
 \end{aligned}$$

Which is simplifies to

$$\frac{dV}{dt} = -(H - H^*)^2 \gamma - (H - H^*) (V - V^*) - (H - H^*) (V - V^*) \Gamma \left(\frac{\alpha_1}{1 + b_1 H} \right) - (V - V^*)^2 - (V - V^*) (P - P^*) \left(\frac{\alpha_2}{1 + b_2 V} \right)$$

hus, $\frac{dV}{dt}$ is a quadratic form which can be

expressed as $\frac{dV}{dt} = -X^T A X$ where

$X^T = (H - H^*, V - V^*, P - P^*)$ and A is an

asymmetric matrix given by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{8}$$

where: $a_{11} = \gamma$, $a_{12} = 1$, $a_{13} = \frac{\alpha_1}{1 + b_1 H}$, $a_{22} = 1$,

$a_{23} = \frac{\alpha_2}{1 + b_2 V}$ and $a_{33} = 0$

We note that the interior steady state $E_8(H^*, V^*, P^*)$ is globally asymptotically stable

if $\frac{dV}{dt} < 0$, that is matrix A is positive definite as stressed by [9].

Now that, the matrix A in equation (8) is positive if $a_{11} > 0$, $a_{13} > 0$, $a_{12} > 0$, $a_{22} > 0$,

$a_{33} > 0$ and $a_{11} a_{22} - a_{12}^2 > 0$. $a_{11} > 0$ gives

$\gamma > 0$; $a_{13} > 0$ gives $\alpha_1 > 0$; $a_{12} > 0$ gives

$a_{12} = 0$; $a_{22} > 0$ gives $a_{22} = 0$ and

$a_{11} a_{22} - a_{12}^2 > 0$. This complete the proof

4.2 Existence of the Hopf-bifurcation in the model

In this section, we observed that the dynamical system (1) have eight steady states. But, we will now study the dynamical behavior of the system about the interior steady state $E_8 \left(\frac{\alpha_1 H}{1 + b_1 H}, \frac{\alpha_2 P}{1 + b_2 V}, P^* \right)$.

The Jacobian matrix of the system (1) is given below:

$$J(E_8) = \begin{pmatrix} m_1 & m_6 & 0 \\ m_4 & m_2 & m_7 \\ 0 & m_5 & m_3 \end{pmatrix} \tag{9}$$

The characteristics polynomial is given by

$$\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0 \tag{10}$$

Where:

$$A_1 = -m_1 - m_2 - m_3,$$

$$A_2 = m_3 (m_1 + m_2) + m_1 m_2 - m_4 m_6 - m_5 m_7,$$

$$A_3 = m_5 m_7 (m_1 + m_2) - m_3 (m_1 m_2 - m_4 m_6) - m_2 m_5 m_7$$

$$m_1 = \gamma - \frac{2H^* \gamma}{k} + \frac{\alpha_1 H^* V^* b_1}{1 + b_1 H^*} - \frac{\alpha_1 V^*}{1 + b_1 H^*} - \mu_1 - \delta,$$

$$m_2 = \frac{\alpha_2 V^* P^* b_2}{(1 + b_2 V^*)^2} + \frac{\alpha_2 P^*}{1 + b_2 V^*} - \mu_2,$$

$$m_3 = \frac{\alpha_2 V^*}{1 + b_2 V^*} - \mu_3,$$

$$m_4 = -\frac{\alpha_1 H^* V^* b_1}{(1 + b_1 H^*)^2} + \frac{\alpha_1 V^*}{1 + b_1 H^*},$$

$$m_5 = \frac{\alpha_2 V^* P^* b_2}{(1 + b_2 V^*)^2} + \frac{\alpha_2 P^*}{1 + b_2 V^*},$$

$$m_6 = \frac{-\alpha_1 H^*}{1 + b_1 H^*} \text{ and } m_7 = -\frac{\alpha_2 V^*}{1 + b_2 V^*}.$$

The Routh-Hurwitz criteria gives a necessary and sufficient conditions for all the roots of the dynamical system of (1) to have a negative real part and which are as follows: $A_1 > 0$, $A_2 > 0$,

$A_3 > 0$ and $A_1 A_2 > A_3$. From the Routh-Hurwitz conditions, it is however difficult to interpret the results in ecological term. Hence, if one of the conditions stated by Routh-Hurwitz is violated, then, the dynamical system would be unstable around the positive interior steady state $E_8(H^*, V^*, P^*)$.

We therefore, study the Hopf-bifurcation of the dynamical system of (1) by method of [19]. This is done by taking γ as the bifurcation parameter.

Now the necessary and sufficient condition of the

existence of Hopf-bifurcation, if it exist is $\gamma = \gamma_0$ such that:

- i. $A_1(\gamma_0) > 0$
- ii. $A_2(\gamma_0) > 0$
- iii. $A_1(\gamma_0)A_2(\gamma_0) > A_3(\gamma_0)$
- iv. Let consider the eigenvalues of the characteristics polynomial (10) is of the form $\lambda_i = u_i \pm v_i$, then $\frac{du_i}{d\gamma} \neq 0, i = 1, 2, 3$

After substituting the values in the condition (iii) $A_1A_2 > A_3$ becomes:

$$(-m_1 - m_2 - m_3)\gamma[m_3(m_1 + m_2) + m_1m_2 - m_4m_6 - m_5m_7]\gamma - [m_5m_7(m_1 + m_2) - m_3(m_1m_2 - m_4m_6) - m_2m_5m_7]\gamma \quad (11)$$

Equation (11) is rewritten as

$$n_1n_2\gamma^2 - n_3\gamma = 0 \quad (12)$$

Equation (12) is rewrites as

$$D_1\gamma^2 - D_2\gamma = 0 \quad (13)$$

Where: $D_1 = n_1n_2, D_2 = n_3,$

$$n_1 = (-m_1 - m_2 - m_3),$$

$$n_2 = m_3(m_1 + m_2) + m_1m_2 - m_4m_6 - m_5m_7 \text{ and}$$

$$n_3 = m_5m_7(m_1 + m_2) - m_3(m_1m_2 - m_4m_6) - m_2m_5m_7$$

Therefore, equation (13) has atleast one positive root say $\gamma = \gamma_0$. Hence, one pair of the eigenvalues of the characteristics polynomial at $\gamma = \gamma_0$ are of the form $\lambda = \pm iv$ where v is a positive real number.

In a mean time, we verify the Hopf-bifurcation condition (iii) by setting $\lambda = u \pm iv$ in equation (13) by separating real and imaginary parts, we have:

$$(u \pm iv)^3 + A_1(u \pm iv)^2 + A_2(u \pm iv) + A_3 = 0 \quad (14)$$

Equation (14) is resolve and separated into real and imaginary parts as given below:

$$u^3 + A_1u^2 + (A_2 - 3v^2)u + (A_3 - A_1v^2) = 0 \quad (15)$$

$$-v^3 + (3u^2 + 2A_1u + A_2)v = 0 \quad (16)$$

Substituting the values of v^2 from equation (15) into equation (16), we have:

$$u^3 + A_1u^2 + (A_2 - 3f(u))u + (A_3 - A_1f(u)) = 0 \quad (17)$$

Where, $f(u) = (3u^2 + 2A_1u + A_2)$ from equation (17), differentiating it with respect to γ by setting $\gamma = \gamma_0$, we have:

$$\left. \frac{du}{d\gamma} \right|_{\gamma=\gamma_0} = \frac{f(u) \frac{dA_1}{d\gamma_0} - \frac{dA_3}{d\gamma_0}}{3f(0) + A_1f'(0) + A_2} \neq 0, \quad \text{Since,}$$

$$f(u) \frac{dA_1}{d\gamma} - \frac{dA_3}{d\gamma} = -f(u) \left(1 - \frac{2H^*}{k} \right) - \left[m_5m_7 \left(1 - \frac{2H^*}{k} \right) - m_2m_5 \left(1 - \frac{2H^*}{k} \right) \right] \left(\frac{k - 2H^*}{k} \right) [(m_2m_3 - m_5m_7) - f(u)] \neq 0$$

This ensures that the dynamical system has a Hopf-bifurcation around the interior steady state E_8 . Therefore, the intrinsic growth rate of honeybee crosses its threshold value $\gamma = \gamma_0$. Then, all the populations begin to oscillate around the interior steady state

4.3 Numerical results for the model

Table 1: State Variables and Parameters

Symbol	Value	Source
$H(t)$	2000	Assumed
$V(t)$	200-1000	Assumed
$P(t)$	0-100	Assumed
γ	0.6	[20]
k	2500	[13]
α_1	0.5	[20]
α_2	0.5	[20]
b_1	0.0025	[8]
b_2	0.003	[8]
μ_1	0.0001	[8]
μ_2	0.0001	[8]
μ_3	0.0001	[8]
η_1	0.125	[8]
η_2	0.125	[8]
δ	0.3	[13]

Experiment 1: The impact of Infestation Coefficient of *Varroa-mite* on Honeybee population

In this experiment, impact of infestation coefficient of *Varroa-mite* on honeybee is studied by considering different values: $\alpha_1 = 0.2, \alpha_1 = 0.4, \alpha_1 = 0.6$ and $\alpha_1 = 0.8$. The parameter values in Table 1 were used for this experiment with an initial condition of $H(0) = 2000$ and $V(0) = 100$. Hence, the numerical results are as follows:

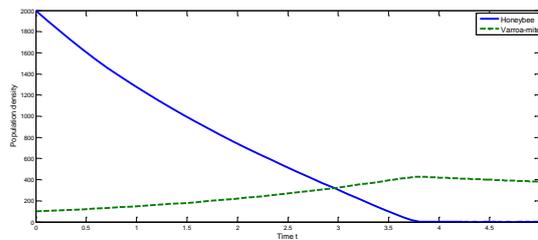


Figure 1: Impact of infestation coefficient of *Varroa-mite* on honeybee at $\alpha_1 = 0.2$

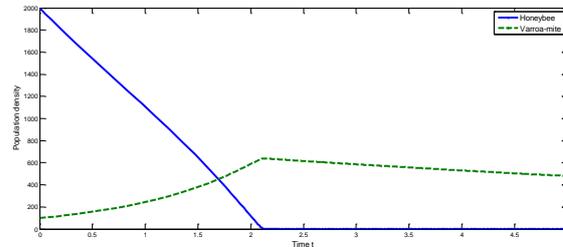


Figure 2: Impact of infestation coefficient of *Varroa-mite* on honeybee at $\alpha_1 = 0.4$

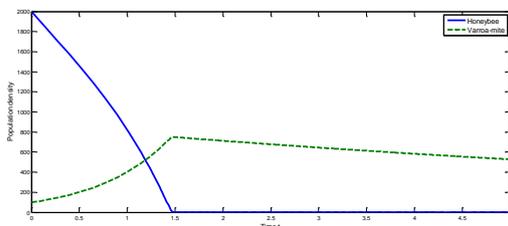


Figure 3: Impact of infestation coefficient of *Varroa-mite* on honeybee at $\alpha_1 = 0.6$

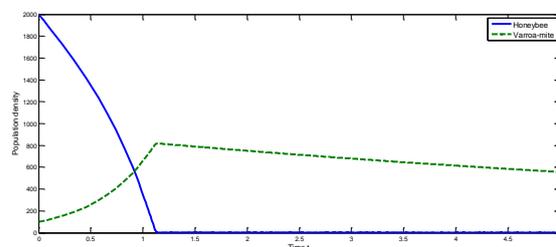


Figure 4: Impact of infestation coefficient of *Varroa-mite* on honeybee at $\alpha_1 = 0.8$

In this experiment, figure 1 through figure 4 indicated that impact of infestation coefficient of *Varroa-mite* on honeybee population. The results have shown that, *Varroa-mite* is able to drive honeybee population to extinction in a relatively small time with higher infestation coefficient as illustrated in figure 4.

Experiment 2: Impact of Capture Coefficient of Pseudoscorpion on Varroa-mite

In this experiment, impact of capture coefficient of *Pseudoscorpion* on *Varroa-mite* is studied by considering different values: $\alpha_2 = 0.1$, $\alpha_2 = 0.2$, $\alpha_2 = 0.4$, $\alpha_2 = 0.6$ and $\alpha_2 = 0.8$. The parameter values in table 1 were used for this experiment with an initial

condition: $H(0) = 2000$, $V(0) = 1,000$ and $P(0) = 100$. Hence, the numerical results are as follows:

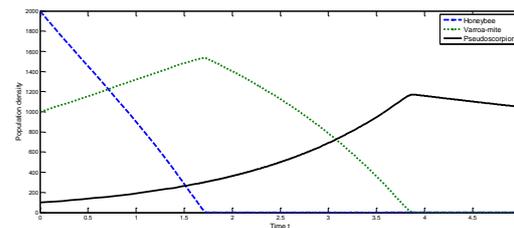


Figure 5: Impact of capture coefficient of *Pseudoscorpion* on *Varroa-mite* at $\alpha_2 = 0.1$

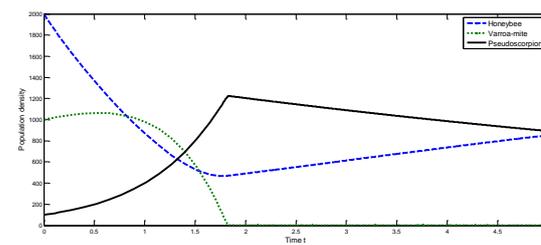


Figure 6: Impact of capture coefficient of *Pseudoscorpion* on *Varroa-mite* at $\alpha_2 = 0.2$

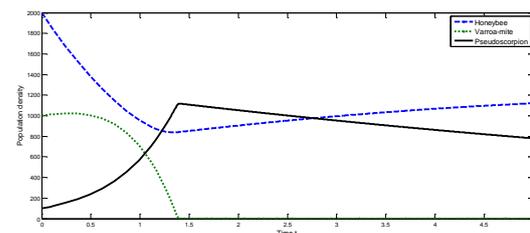


Figure 7: Impact of capture coefficient of *Pseudoscorpion* on *Varroa-mite* at $\alpha_2 = 0.4$

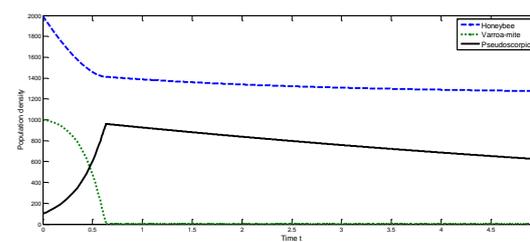


Figure 8: Impact of capture coefficient of *Pseudoscorpion* on *Varroa-mite* at $\alpha_2 = 0.6$

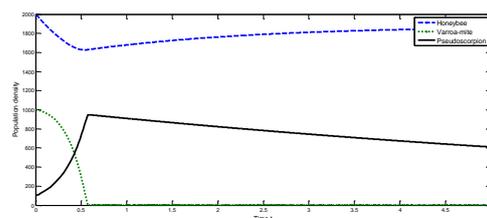


Figure 9: Impact of capture coefficient of *Pseudoscorpion* on *Varroa-mite* at $\alpha_2 = 0.8$

Figure 5 through figure 9 is depicting the role of capture coefficient of *Pseudoscorpion* on *Varroa-mite* population in the presence of honeybee colony. Results shown from these figures indicated that, higher capture coefficient of *Pseudoscorpion* (figure 9) on *Varroa-mite* has eradicated the *Varroa-mite* population within a shortest possible time (6 months) and boosted up the population of honeybee thereafter.

Experiment 3: Impact of Virus Infection on Honeybee Population

Since the dynamical system (1) portrays ecological interaction of multispecies, where one of the species is a vector of pathogen, then, it is necessary to study the impact of the virus on the dynamics of the affected specie (honeybee). The parameter values of table 1 is used alongside an initial value of honeybee as $H(0) = 2,000$. The virus infection parameter δ is varied as follows: $\delta = 0$, $\delta = 0.2$, $\delta = 0.4$, $\delta = 0.6$, $\delta = 0.8$ and $\delta = 0.9$. Thus, the numerical result is given below:

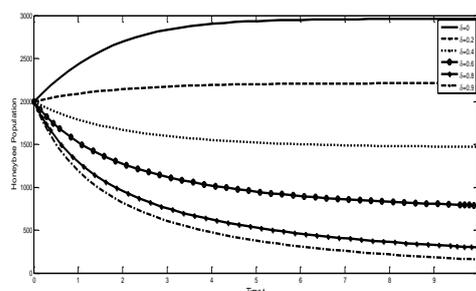


Figure 10: Impact of different virus infection rate on honeybee population.

The simulation results has shown the negative impact of virus invasion on the honeybee depicted in figure 10. Furthermore, as δ is varied while other parameter values are hold constant; indicated a continuous decline in the honeybee population. Hence, Honeybee population declines as the result of virus attack. With larger numbers of *Varroa-mites* in a colony, a greater proportion of honeybees is affected. Thus, without treatment a colony of honeybee infested with *Varroa-mite* dies within one to three years ([21], [22], [23] and [24]).

5. Conclusion

A mathematical model for the dynamics of honeybee, *Varroa-mite* and *Pseudoscorpion* in prey-predator interaction in which the predators show Holing Type II functional response to their prey with honeybee infested with virus-carrying *Varroa-mite* was proposed and analysed. The study investigated the global stability of the interior steady state and the existence of hofb bifurcation of the model. Obviously, the

numerical results portrayed that honeybee population decline rapidly with higher infestation rate of virus-carrying *Varroa-mite*. However, the honeybee population stabilizes sharply with smaller virus infection rate alongside much higher capture coefficient of *Pseudoscorpion* on *Varroa-mite*. Thus, introducing *Pseudoscorpion* as a super predator on *Varroa-mite* in the system significantly breaks honeybee from heavy pressure and population decline.

Conflict of Interest

The author declares that there is no conflict of interest

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